Appendix to Marginalized Exponential Random Graph Models published in the Journal of Computational and Graphical Statistics

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Derivation of Likelihood Equation

First we express the mean of $\mathbf{Y} = (Y_{12}, Y_{13}, \ldots, Y_{n-1,n})'$ as

$$\boldsymbol{\pi} = \mathbb{E}\mathbf{Y} = \sum_{\mathbf{y}\in S} \mathbf{y} \exp\left(\boldsymbol{\Psi}'\mathbf{y} + \boldsymbol{\eta}(\boldsymbol{\theta})'\mathbf{Z}(\mathbf{y}) - \kappa\left\{\boldsymbol{\eta}(\boldsymbol{\theta}, \boldsymbol{\Psi})\right\}\right) = \partial\kappa/\partial\boldsymbol{\Psi}$$

and that of $\mathbf{Z}$ by

$$\mathbb{E}\mathbf{Z}(\mathbf{Y}) = \sum_{\mathbf{y}\in S} \mathbf{Z}(\mathbf{y}) \exp\left(\boldsymbol{\Psi}'\mathbf{y} + \boldsymbol{\eta}(\boldsymbol{\theta})'\mathbf{Z}(\mathbf{y}) - \kappa\left\{\boldsymbol{\eta}(\boldsymbol{\theta}, \boldsymbol{\Psi})\right\}\right) = \partial\kappa/\partial\boldsymbol{\eta}.$$

Now we obtain

$$\partial\boldsymbol{\pi}/\partial\boldsymbol{\Psi} = \left[\sum_{\mathbf{y}\in S} \mathbf{y}\mathbf{y}' \exp\left(\boldsymbol{\Psi}'\mathbf{y} + \boldsymbol{\eta}(\boldsymbol{\theta})'\mathbf{Z}(\mathbf{y}) - \kappa\left\{\boldsymbol{\eta}(\boldsymbol{\theta}, \boldsymbol{\Psi})\right\}\right)\right] - \boldsymbol{\pi}\boldsymbol{\pi}' = \mathbf{C}_{\mathbf{Y}},$$

$$\partial\boldsymbol{\pi}/\partial\boldsymbol{\eta} = \left[\sum_{\mathbf{y}\in Y} \mathbf{y}\mathbf{Z}(\mathbf{y})' \exp\left(\boldsymbol{\Psi}'\mathbf{y} + \boldsymbol{\eta}(\boldsymbol{\theta})'\mathbf{Z}(\mathbf{y}) - \kappa\left\{\boldsymbol{\eta}(\boldsymbol{\theta}, \boldsymbol{\Psi})\right\}\right)\right] - \boldsymbol{\pi}\mathbb{E}\mathbf{Z}(\mathbf{y})' = \mathbf{C}_{\mathbf{Y},\mathbf{Z}}$$

and

$$\partial\mathbb{E}Z/\partial\boldsymbol{\eta} = \left[\sum_{\mathbf{y}\in S} \mathbf{Z}(\mathbf{y})\mathbf{Z}(\mathbf{y})' \exp\left(\boldsymbol{\Psi}'\mathbf{y} + \boldsymbol{\eta}(\boldsymbol{\theta})'\mathbf{Z}(\mathbf{y}) - \kappa\left\{\boldsymbol{\eta}(\boldsymbol{\theta}, \boldsymbol{\Psi})\right\}\right)\right] - \mathbb{E}\mathbf{Z}(\mathbf{y})\mathbb{E}\mathbf{Z}(\mathbf{y})' = \mathbf{C}_{\mathbf{Z}}.$$

We have

$$\begin{pmatrix} \partial l/\partial\boldsymbol{\Psi} \\ \partial l/\partial\boldsymbol{\eta} \end{pmatrix} = \begin{pmatrix} \mathbf{y}^{obs} - \mathbb{E}\mathbf{Y} \\ \mathbf{z}^{obs} - \mathbb{E}\mathbf{Z} \end{pmatrix} = \begin{pmatrix} \partial\boldsymbol{\pi}/\partial\boldsymbol{\Psi} & \partial\boldsymbol{\eta}/\partial\boldsymbol{\Psi} \\ \partial\boldsymbol{\pi}/\partial\boldsymbol{\eta} & \partial\boldsymbol{\eta}/\partial\boldsymbol{\eta} \end{pmatrix} \begin{pmatrix} \partial l/\partial\boldsymbol{\pi} \\ \partial l/\partial\boldsymbol{\eta} \end{pmatrix} = \begin{pmatrix} \mathbf{C}_{\mathbf{Y}} & \mathbf{0} \\ \mathbf{C}_{\mathbf{Z},\mathbf{Y}} & \mathbf{I} \end{pmatrix} \begin{pmatrix} \partial l/\partial\boldsymbol{\pi} \\ \partial l/\partial\boldsymbol{\eta} \end{pmatrix}.$$

Therefore

$$\begin{pmatrix} \partial l/\partial\boldsymbol{\pi} \\ \partial l/\partial\boldsymbol{\eta} \end{pmatrix} = \begin{pmatrix} \mathbf{C}_{\mathbf{Y}} & \mathbf{0} \\ \mathbf{C}_{\mathbf{Z},\mathbf{Y}} & \mathbf{I} \end{pmatrix}^{-1} \begin{pmatrix} \mathbf{y}^{obs} - \mathbb{E}\mathbf{Y} \\ \mathbf{z}^{obs} - \mathbb{E}\mathbf{Z} \end{pmatrix} = \begin{pmatrix} \mathbf{C}_{\mathbf{Y}}^{-1} & \mathbf{0} \\ -\mathbf{C}_{\mathbf{Z},\mathbf{Y}}\mathbf{C}_{\mathbf{Y}}^{-1} & \mathbf{I} \end{pmatrix} \begin{pmatrix} \mathbf{y}^{obs} - \mathbb{E}\mathbf{Y} \\ \mathbf{z}^{obs} - \mathbb{E}\mathbf{Z} \end{pmatrix}.$$

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Finally we derive

$$
\begin{aligned}
\begin{pmatrix} \partial l/\partial \boldsymbol{\beta} \\ \partial l/\partial \boldsymbol{\theta} \end{pmatrix} &= \begin{pmatrix} \partial \boldsymbol{\pi}/\partial \boldsymbol{\beta} & \partial \boldsymbol{\eta}/\partial \boldsymbol{\beta} \\ \partial \boldsymbol{\pi}/\partial \boldsymbol{\theta} & \partial \boldsymbol{\eta}/\partial \boldsymbol{\theta} \end{pmatrix} \begin{pmatrix} \partial l/\partial \boldsymbol{\pi} \\ \partial l/\partial \boldsymbol{\eta} \end{pmatrix} \\
&= \begin{pmatrix} \mathbf{X}_f' \left(\partial \boldsymbol{\pi}/\partial \boldsymbol{\nu}\right) & \mathbf{0} \\ \mathbf{0} & \triangledown \boldsymbol{\eta}(\boldsymbol{\theta})' \end{pmatrix} \begin{pmatrix} \mathbf{C}_{\mathbf{Y}}^{-1} & \mathbf{0} \\ -\mathbf{C}_{\mathbf{Z},\mathbf{Y}}\mathbf{C}_{\mathbf{Y}}^{-1} & \mathbf{I} \end{pmatrix} \begin{pmatrix} \mathbf{y}^{obs} - \mathbb{E}\mathbf{Y} \\ \mathbf{z}^{obs} - \mathbb{E}\mathbf{Z} \end{pmatrix},
\end{aligned}
$$

where $\partial \boldsymbol{\pi}/\partial \boldsymbol{\nu} := \mathbf{D}$ yielding

$$
\begin{pmatrix} \partial l/\partial \boldsymbol{\beta} \\ \partial l/\partial \boldsymbol{\theta} \end{pmatrix} = \begin{pmatrix} \mathbf{X}_f' \mathbf{D} \mathbf{C}_{\mathbf{Y}}^{-1}(\mathbf{y}^{obs} - \boldsymbol{\pi}) \\ \triangledown \boldsymbol{\eta}(\boldsymbol{\theta})' \left\{ -\mathbf{C}_{\mathbf{Z},\mathbf{Y}}\mathbf{C}_{\mathbf{Y}}^{-1}(\mathbf{y}^{obs} - \boldsymbol{\pi}) + \mathbf{z}^{obs} - \mathbb{E}\mathbf{Z} \right\} \end{pmatrix}.
$$