1. Introduction. In 1972, Gary H. Meisters and Wolfgang M. Schmidt [4] proved an elegant and remarkable result concerning Fourier analysis on the circle group $\mathbb{T}$, which is the set of complex numbers of modulus 1. We denote the normalised Haar measure on $\mathbb{T}$ by $\mu$. That is, for an integrable, complex-valued function $f$ on $\mathbb{T}$,

$$\int_{\mathbb{T}} f \, d\mu = \frac{1}{2\pi} \int_{0}^{2\pi} f(e^{it}) \, dt.$$ 

The space $L^2(\mathbb{T})$ is the family of functions $f$ on $\mathbb{T}$ such that $\int_{\mathbb{T}} |f|^2 \, d\mu < \infty$.

Meisters and Schmidt proved that if a function $f$ in $L^2(\mathbb{T})$ has the property that $\int_{\mathbb{T}} f \, d\mu = 0$, then $f$ is of the form $\sum_{j=1}^{3} (f_j - \delta_{b_j} \ast f_j)$, for suitable $f_1, f_2, f_3$ in $L^2(\mathbb{T})$ and $b_1, b_2, b_3$ in $\mathbb{T}$, where $\delta_b$ denotes the Dirac measure at $b$ and $\ast$ denotes convolution. Note that $\delta_b \ast f$ is the function $z \mapsto -f(b^{-1}z)$ on $\mathbb{T}$ and is called the translation of $f$ by $b$. They deduced that every linear form $T$ on $L^2(\mathbb{T})$ such that $T(\delta_a \ast f) = T(f)$ for all $a \in \mathbb{T}$ and all $f$ in $L^2(\mathbb{T})$ must be continuous. These results of Meisters and Schmidt were generalized further on $\mathbb{T}$ in [6], and to the non-compact case of $\mathbb{R}$ in [6, 7].

Meisters and Schmidt actually presented their results in the context of compact abelian groups and spaces of abstract distributions. The present purpose is to give a more accessible proof in the case of the circle group and $L^2(\mathbb{T})$ only, with the intention as well of incorporating the more general results in [6]. The motivation for doing this was research work carried out recently with Susumu Okada [9], which makes central use of the main result presented here.

The presentation is based on classical measure theory and the Fourier transform in the space $L^2(\mathbb{T})$ of the circle group $\mathbb{T}$. The techniques are those of Meisters and Schmidt in all essentials, although they are used to obtain a more general result. As mentioned, $\mathbb{T}$ consists of all numbers $z \in \mathbb{C}$ such that $|z| = 1$. We can identify $\mathbb{T}$ with the interval $[0, 2\pi)$ by means of the one-to-one and onto mapping $x \mapsto e^{ix}$ from $[0, 2\pi)$ to $\mathbb{T}$. Statements about $\mathbb{T}$ may be expressed as equivalent statements about the interval $[0, 2\pi)$. Suitable sources for background knowledge in integration theory and Fourier analysis may be found in [1, 2, 10], for example. The discussion also makes use of the integration of functions in multivariate analysis. The symbols $\mathbb{N}, \mathbb{Z}, \mathbb{R}$ and $\mathbb{C}$, respectively denote the sets consisting of the positive integers, the integers, the real numbers and the complex numbers.

2. Notation and statement of the main theorem. Let $M(\mathbb{T})$ denote the set of regular Borel measures on $\mathbb{T}$. Then, $M(\mathbb{T})$ has an operation of convolution, which we denote by $\ast$. If $\lambda \in M(\mathbb{T})$, $\lambda^n$ denotes the measure $\lambda \ast \lambda \ast \cdots \ast \lambda$, where $\lambda$ appears $n$ times. If $b \in \mathbb{T}$,
\[ \delta_b \in M(\mathbb{T}) \] denotes the Dirac measure at \( b \). As above, the normalised Haar measure on \( \mathbb{T} \) is denoted by \( \mu \), and \( \mu \in M(\mathbb{T}) \). If \( \lambda \in M(\mathbb{T}) \) and \( f \in L^2(\mathbb{T}) \), then \( f * \lambda \in L^2(\mathbb{T}) \) and

\[ f * \lambda(x) = \int_{\mathbb{T}} f(z^{-1}x) d\lambda(z), \]

for almost all \( x \in \mathbb{T} \).

If \( s \in \mathbb{N} \) and \( b \in \mathbb{T} \) put

\[ \mu_{b,s} = (\delta_1 - \delta_b)^s. \]

A function in \( L^2(\mathbb{T}) \) is called a difference of order \( s \) if it is of the form

\[ \mu_{b,s} * g, \]

for some \( g \in L^2(\mathbb{T}) \) and some \( b \in \mathbb{T} \). The vector space of functions that are finite sums of differences of order \( s \) is denoted by \( D_s(L^2(\mathbb{T})) \). Note that a difference of order 1 is a function of the form \( g - \delta_b * g \) for some \( b \in \mathbb{T} \) and \( g \in L^2(\mathbb{T}) \). Also, a difference of order 2 is a function of the form \( g - 2^{-1}(\delta_b + \delta_{-b}) * g \) for some \( b \in \mathbb{T} \) and \( g \in L^2(\mathbb{T}) \).

**THE MAIN THEOREM.** Let \( s \in \mathbb{N} \). Then, \( D_s(L^2(\mathbb{T})) \) is independent of \( s \) and, in fact,

\[ D_s(L^2(\mathbb{T})) = \left\{ f : f \in L^2(\mathbb{T}) \text{ and } \int_{\mathbb{T}} f \, d\mu = 0 \right\}. \]

Also, if \( f \in D_s(L^2(\mathbb{T})) \), \( f \) is equal to a sum of \( 2s + 1 \) differences of order \( s \).

The theorem was proved in the case of \( s = 1 \) by G. Meisters and W. Schmidt [1]. Thus, they proved: a function in \( L^2(\mathbb{T}) \) is a sum of three differences of order 1 if and only if \( \int_{\mathbb{T}} f \, d\mu = 0 \). An immediate consequence of this is: if \( T \) is a linear form on \( L^2(\mathbb{T}) \) such that \( T(\delta_b * f) = T(f) \) for all \( f \in L^2(\mathbb{T}) \) and all \( b \in \mathbb{T} \), then \( T \) is a multiple of the Haar measure on \( \mathbb{T} \) and \( T \) is therefore continuous. However, Meisters showed in [5] that the corresponding statement fails when the non-compact group \( \mathbb{R} \) replaces the compact group \( \mathbb{T} \).

3. The Fourier transform. The Fourier transform for the circle group is an essential concept in the proof of The Main Theorem. If \( \lambda \in M(\mathbb{T}) \), the Fourier transform of \( \lambda \) is the function \( \hat{\lambda} \) on \( \mathbb{Z} \) given by

\[ \hat{\lambda}(n) = \int_{\mathbb{T}} z^{-n} d\lambda(z), \]

for \( n \in \mathbb{Z} \).

Now \( L^2(\mathbb{T}) \subseteq M(\mathbb{T}) \), and the Fourier transform is a Hilbert space isometry from \( L^2(\mathbb{T}) \) onto \( \ell^2(\mathbb{Z}) \). If \( \lambda, \nu \in M(\mathbb{T}) \), \( (\lambda * \nu) = \hat{\lambda} \hat{\nu} \). If \( b \in \mathbb{T} \), \( \hat{\delta}_b(n) = b^{-n} \). Note that the condition on \( f \) in The Main Theorem above, namely that \( \int_{\mathbb{T}} f \, d\mu = 0 \), is equivalent to the condition \( \hat{f}(0) = 0 \).  

4. Preliminary results. The main obstacle to proceeding is that we have no way of deciding whether a function is a finite sum of differences of order \( s \). The way out is given by a result of Meisters and Schmidt [4], which changes the problem into a question of whether certain integrals are convergent or divergent. We then obtain estimates on these integrals to obtain our results. Our approach here is the same as in [4], the main difference being that we consider differences of order \( s \), not only \( s = 1 \), and a corresponding larger class of integrals.

In places like equation (1) below, and elsewhere, formal expressions of the form 0/0 and \( x/0 \) for some \( x > 0 \) may occur. We take 0/0 to be 0 and \( x/0 \) to be \( \infty \) is such cases.
Theorem 1. Let \( f \in L^2(\mathbb{T}) \), let \( r \in \mathbb{N} \) and let \( \mu_1, \mu_2, \ldots, \mu_r \in M(\mathbb{T}) \). Then the following statements hold. If
\[
\sum_{n=-\infty}^{\infty} \frac{|\hat{f}(n)|^2}{\sum_{j=1}^{r} |\hat{\mu}_j(n)|^2} < \infty,
\]
then \( f = \sum_{j=1}^{r} \mu_j \ast f_j \), for some \( f_1, f_2, \ldots, f_r \in L^2(\mathbb{T}) \). The converse statement is also true.

Proof. Let \( f \in L^2(\mathbb{T}) \), let \( r \in \mathbb{N} \) and let \( \mu_1, \mu_2, \ldots, \mu_r \in M(\mathbb{T}) \) be such that (1) holds. Then define
\[ h(n) = \max \left\{ |\hat{\mu}_1(n)|, |\hat{\mu}_2(n)|, \ldots, |\hat{\mu}_r(n)| \right\}, \]
for each \( n \in \mathbb{Z} \). Then define subsets \( B_0, B_1, \ldots, B_r \) of \( \mathbb{Z} \), as follows: we put
\[ B_0 = \left\{ n : n \in \mathbb{Z} \text{ and } h(n) = 0 \right\}, \]
and then
\[ B_j = \left\{ n : n \in \mathbb{Z}, n \notin \bigcup_{k=0}^{j-1} B_k \text{ and } h(n) = |\hat{\mu}_j(n)| \right\}, \]
for \( j = 1, 2, \ldots, r \). The sets \( B_0, B_1, \ldots, B_r \) are disjoint and their union is \( \mathbb{Z} \). Note that if \( n \in B_j \) for some \( j \in \{1, 2, \ldots, r\} \), then \( |\hat{\mu}_j(n)| = h(n) > 0 \).

Thus, for \( j \in \{1, 2, \ldots, r\} \), a complex-valued function \( h_j \) may be defined on \( \mathbb{Z} \) by
\[ h_j(n) = \begin{cases} \hat{f}(n) / \hat{\mu}_j(n), & \text{if } n \in B_j; \\ 0, & \text{if } x \notin B_j. \end{cases} \]

Then, for \( j \in \{1, 2, \ldots, r\} \),
\[
\sum_{n \in \mathbb{Z}} |h_j(n)|^2 = \sum_{n \in B_j} \frac{|\hat{f}(n)|^2}{|\hat{\mu}_j(n)|^2},
\]
\[
= r \sum_{n \in \mathbb{Z}} \frac{\max\{|\hat{\mu}_1(n)|^2, \ldots, |\hat{\mu}_r(n)|^2\}}{r \sum_{n \in \mathbb{Z}} \sum_{j=1}^{r} |\hat{\mu}_j(n)|^2},
\]
\[
\leq r \sum_{n \in \mathbb{Z}} \sum_{j=1}^{r} |\hat{\mu}_j(n)|^2,
\]
\[
< \infty.
\]

Hence \( h_j \in \ell^2(\mathbb{Z}) \). By the Riesz-Fischer Theorem, there is \( f_j \in L^2(\mathbb{R}) \) such that \( \hat{f}_j = h_j \), for all \( j = 1, 2, \ldots, r \). The definition of \( h_j \) gives
\[ \hat{f}(n) = \hat{\mu}_j(n) h_j(n) = \hat{\mu}_j(n) \hat{f}_j(n), \]
for all \( n \in B_j \). Now, note that \( B_0, B_1, \ldots, B_r \) are disjoint and have union equal to \( \mathbb{Z} \). Also, \( \hat{f}_j = 0 \) on \( B_k \) if \( j \neq k \), and \( \hat{f} = 0 \) on \( B_0 \). It follows that, on \( \mathbb{Z} \),
\[ \hat{f} = \sum_{j=1}^{r} \hat{\mu}_j \hat{f}_j = \sum_{j=1}^{r} (\mu_j \ast f_j) = \left( \sum_{j=1}^{r} \mu_j \ast f_j \right) \hat{\mu}_j \hat{f}_j. \]
As functions that have equal Fourier transforms must be equal, we have the required identity

\[ f = \sum_{j=1}^{r} \mu_j * f_j. \]

As mentioned above, the converse statement in Theorem 1 is true, but it is not needed here and is not proved (for further details see [4, pages 411-412] and [6, pages 77-78]).

**Lemma 1.** If \( x \in [0, \pi] \),

\[ \frac{x}{\pi} \leq \sin \frac{x}{2}. \]

**Proof.** Observe that \( x \mapsto \sin x/x \) is decreasing on \((0, \pi/2]\). So, we see that for all \( 0 \leq x \leq \pi \),

\[ \sin \frac{x}{2} \geq \frac{x}{2} \frac{\pi}{2} = \frac{x}{\pi}. \]

**Lemma 2.** Let \( s, m \in \mathbb{N} \) with \( m > 2s \). Then we have:

\[ \int_{(0,1)^m} \frac{dx_1 dx_2 \ldots dx_m}{\sum_{j=1}^{m} x_j^{2s}} < \infty \text{ and } \int_{(0,2\pi)^m} \frac{dx_1 dx_2 \ldots dx_m}{\sum_{j=1}^{m} \sin^{2s}(x_j/2)} < \infty. \]

**Proof.** Let \( m > 2s \), let \( S^{m-1} \) denote the surface of the unit sphere in \( \mathbb{R}^m \), and let \( \sigma_{m-1} \) denote the usual surface measure on \( S^{m-1} \). Observe that if \((x_1, x_2, \ldots, x_m) \in [0, 1)^m\), then

\[ \left( \sum_{j=1}^{m} x_j^{2} \right)^{1/2} \leq \sqrt{m}. \]

Also, there is \( c > 0 \) such that

\[ \left( \sum_{j=1}^{m} x_j^{2} \right)^{s} \leq c \cdot \sum_{j=1}^{m} x_j^{2s}, \text{ for all } (x_1, x_2, \ldots, x_m) \in \mathbb{R}^m. \]

Thus, we have

\[ \int_{(0,1)^m} \frac{dx_1 dx_2 \ldots dx_m}{\sum_{j=1}^{m} x_j^{2s}} \leq c \int_{(0,1)^m} \frac{dx_1 dx_2 \ldots dx_m}{\left( \sum_{j=1}^{m} x_j^{2} \right)^{s}} \leq c \int_{0}^{\sqrt{m}} \int_{S^{m-1}} \frac{r^{m-1}}{r^{2s}} d\sigma_{m-1} \, dr \]

\[ = c \cdot \sigma_{m-1}(S^{m-1}) \cdot \int_{0}^{\sqrt{m}} r^{m-2s-1} \, dr < \infty, \]

because \( m > 2s \). So we that the first-mentioned integral is finite.
When we come to the second integral, observe that for any function \( f \) on \([0, 2\pi)^m\),

\[
\int_{[0,2\pi)^m} f(x_1, x_2, \ldots, x_m) dx_1 dx_2 \ldots dx_m = \sum_{J_1, J_2, \ldots, J_m} J_1 \times \ldots \times J_m \int_{J_1 \times \ldots \times J_m} f(x_1, x_2, \ldots, x_m) dx_1 dx_2 \ldots dx_m, \tag{3}
\]

where the sum is taken over all \( 2^m \) combinations of intervals \( J_1, J_2, \ldots, J_m \) where for each \( j \), \( J_j = [0, \pi) \) or \( J_j = [\pi, 2\pi) \). Now if \( p + q = m \) we have

\[
\int_{[0,\pi)^p \times [\pi, 2\pi)^q} \frac{dx_1 dx_2 \ldots dx_m}{\sum_{j=1}^m \sin^2(x_j/2)} = \int_{[0,\pi)^p \times [0,\pi)^q} \frac{dx_1 \ldots dx_p du_1 \ldots du_q}{\sum_{j=1}^p \sin^2(x_j/2) + \sum_{j=1}^q \sin^2((2\pi - u_j)/2)}
\]

\[
= \int_{[0,\pi)^{p+q}} \frac{dx_1 \ldots dx_m}{\sum_{j=1}^m \sin^2(x_j/2)}, \text{ using Lemma 1},
\]

\[
= \pi^{m+2s} \int_{[0,1)^m} \frac{dx_1 \ldots dx_m}{\sum_{j=1}^m x_j^{2s}}, \text{ on putting } x_j/\pi \text{ in placed of } x_j,
\]

\[
< \infty, \tag{4}
\]

upon using (2) and as \( m > 2s \). We now see from (3) and (4) that for \( m > 2s \),

\[
\int_{[0,2\pi)^m} \frac{dx_1 dx_2 \ldots dx_m}{\sum_{j=1}^m \sin^2(x_j/2)} \leq 2^m \pi^{m+2s} \int_{[0,1)^m} \frac{dx_1 \ldots dx_m}{\sum_{j=1}^m x_j^{2s}} < \infty. \tag*{\square}
\]

5. Proof of The Main Theorem. We have to show that a function \( f \) in \( L^2(\mathbb{T}) \) is a finite sum of differences of order \( s \) if and only if \( \hat{f}(0) = 0 \). When this holds we have to show we need at most \( 2s + 1 \) differences of order \( s \).

First: the easy part. Let \( f \in \mathcal{D}_s(L^2(\mathbb{T})) \); we prove \( \hat{f}(0) = 0 \). For \( n \in \mathbb{Z} \), \( s \in \mathbb{N} \) and \( b \in \mathbb{T} \),

\[
\hat{\mu}_{b,s}(n) = [(\delta_1 - \delta_b)^s]^\sim(n) = \left((\delta_1 - \delta_b)^\sim(n)\right)^s = (1 - b^{-n})^s. \tag{5}
\]

However, if \( f \) is a sum of \( q \) differences of order \( s \), there are \( b_1, b_2, \ldots, b_q \in \mathbb{T} \) and \( f_1, f_2, \ldots, f_q \in L^2(\mathbb{T}) \) such that \( f = \sum_{j=1}^q \mu_{b_j,s} * f_j \). Then, by (5),

\[
\hat{f}(n) = \sum_{j=1}^q \hat{\mu}_{b_j,s}(n) \hat{f}_j(n) = \sum_{j=1}^q (1 - b_j^{-n})^s \hat{f}_j(n). \tag{6}
\]

Now, putting \( n = 0 \) means that \( 1 - b_j^{-n} = 1 - 1 = 0 \), so we deduce from (6) that \( \hat{f}(0) = 0 \). Thus, if \( f \) is a finite sum of differences of order \( s \), \( \hat{f}(0) = 0 \). Alternatively we can say that \( f \in \mathcal{D}_s(L^2(\mathbb{R})) \) implies that \( \hat{f}(0) = 0 \), as required.

Second: we prove the hard part. We show that if \( f \in L^2(\mathbb{T}) \) and \( \hat{f}(0) = 0 \), then \( f \) is a sum of \( 2s + 1 \) differences of order \( s \). Let \( n \in \mathbb{Z} \) and \( n \neq 0 \).

Observe that if \( m \in \mathbb{N} \), and if \( \mu_m \) is the normalised Haar measure on \( \mathbb{T}^m \) then for all \( n \in \mathbb{Z} \) with \( n \neq 0 \), and for \( f \in L^1(\mathbb{T}^m) \),

\[
\int_{\mathbb{T}^m} f(z_1^n, z_2^n, \ldots, z_m^n) \, d\mu_m(z_1, z_2, \ldots, z_m) = \int_{\mathbb{T}^m} f(z_1, z_2, \ldots, z_m) \, d\mu_m(z_1, z_2, \ldots, z_m).
\]
The reason is that both sides define an invariant integral on $\mathbb{T}^m$ and so they must be equal by the uniqueness of the Haar integral. So, for all $n \in \mathbb{Z}$ with $n \neq 0$, and for $f \in L^1(\mathbb{T}^m)$,
\[
\int_{\mathbb{T}^m} f(z_1^n, z_2^n, \ldots, z_m^n) \, d\mu_m(z_1, z_2, \ldots, z_m) = \frac{1}{(2\pi)^m} \int_{(0,2\pi)^m} f(e^{ix_1}, e^{ix_2}, \ldots, e^{ix_m}) \, dx_1 \, dx_2 \ldots \, dx_m. \quad (7)
\]
We will use this result with
\[
f(z_1, z_2, \ldots, z_m) = \frac{1}{\sum_{j=1}^m |1 - z_j^{-1}|^{2s}}.
\]
NOW, let $n \in \mathbb{Z}$ with $n \neq 0$ be given. We have
\[
\int_{\mathbb{T}^m} \frac{d\mu_m(y_1, y_2, \ldots, y_m)}{\sum_{j=1}^m |\hat{\mu}_{y_j,s}(n)|^2} = \int_{\mathbb{T}^m} \frac{d\mu_m(y_1, y_2, \ldots, y_m)}{\sum_{j=1}^m |1 - y_j^{-n}|^{2s}}
\]
\[
= \int_{\mathbb{T}^m} f(z_1^n, z_2^n, \ldots, z_m^n) \, d\mu_m(z_1, z_2, \ldots, z_m)
\]
\[
= \frac{1}{(2\pi)^m} \int_{(0,2\pi)^m} f(e^{ix_1}, e^{ix_2}, \ldots, e^{ix_m}) \, dx_1 \, dx_2 \ldots \, dx_m, \text{ using (7),}
\]
\[
= \frac{1}{(2\pi)^m} \int_{(0,2\pi)^m} dx_1 \, dx_2 \ldots \, dx_m
\]
\[
= \frac{1}{4} \cdot \frac{1}{(2\pi)^m} \int_{(0,2\pi)^m} \sum_{j=1}^m \sin^{2s}(x_j/2)
\]
\[
< \infty,
\]
using Lemma 2 and the assumption that $m > 2s$. Taking $m = 2s + 1$ we get that for all $n \in \mathbb{Z}$ with $n \neq 0$,
\[
\int_{\mathbb{T}^{2s+1}} \frac{d\mu_{2s+1}(y_1, y_2, \ldots, y_{2s+1})}{\sum_{j=1}^{2s+1} |\hat{\mu}_{y_j,s}(n)|^2} = M < \infty. \quad (8)
\]
Note that $M$ does not depend upon $n$.

Now,
\[
\int_{\mathbb{T}^{2s+1}} \left( \sum_{n=-\infty, n \neq 0}^\infty \frac{|\hat{f}(n)|^2}{\sum_{j=1}^{2s+1} |\hat{\mu}_{y_j,s}(n)|^2} \right) \, d\mu_{2s+1}(y_1, y_2, \ldots, y_{2s+1})
\]
\[
= \sum_{n=-\infty, n \neq 0}^\infty |\hat{f}(n)|^2 \left( \int_{\mathbb{T}^{2s+1}} \frac{d\mu_{2s+1}(y_1, y_2, \ldots, y_{2s+1})}{\sum_{j=1}^{2s+1} |\hat{\mu}_{y_j,s}(n)|^2} \right)
\]
\[
\leq M \sum_{n=-\infty, n \neq 0}^\infty |\hat{f}(n)|^2, \text{ by (8),}
\]
\[
< \infty.
\]
We deduce from this that
\[
\sum_{n=-\infty, n \neq 0}^\infty \frac{|\hat{f}(n)|^2}{\sum_{j=1}^{2s+1} |\hat{\mu}_{y_j,s}(n)|^2} < \infty,
\]
for almost all \((y_1, y_2, \ldots, y_{2s+1})\) in \(\mathbb{T}^{2s+1}\). So, by Theorem 1, for almost all \((y_1, y_2, \ldots, y_{2s+1})\) in \(\mathbb{T}^{2s}\), there are \(g_1, g_2, \ldots, g_{2s+1} \in L^2(\mathbb{T})\) such that

\[
f = \sum_{j=1}^{2s+1} (g_j - \mu_{y_j,s} * g_j).
\]

That is, we have shown that \(f\) is a sum of \(2s + 1\) differences of order \(s\). \(\square\)

Note that the statement in (9), in that it holds for almost all \((b_1, b_2, \ldots, b_{2s+1}) \in \mathbb{T}^{2s+1}\), has proved more than the original statement in The Main Theorem.

Summarizing, we have the following. Let \(s \in \mathbb{N}\) and \(f \in L^2(\mathbb{T})\). Then, \(\hat{f}(0) = 0\) if and only if \(f\) is a sum of \(2s + 1\) differences of order \(s\). In this case, for almost all \((b_1, b_2, \ldots, b_{2s+1}) \in \mathbb{T}^{2s+1}\), there are \(f_1, f_2, \ldots, f_{2s+1} \in L^2(\mathbb{T})\) such that

\[
f = \sum_{j=1}^{2s+1} (\delta_j - \delta_{b_j}) * f_j.
\]

6. Comments and conclusion. The main result mentioned here has analogues on other compact abelian groups apart from \(\mathbb{T}\) (see [3, 4, 6]). There are also analogous results for non-compact groups (see [6, 7]). There is also the question as to the sharpness of the result in The Main Theorem: if \(\hat{f}(0) = 0\) do we need \(2s + 1\) differences in general in order to express \(f\) as sum of differences of order \(s\), or will fewer suffice? For the space \(\mathcal{D}_1(\mathbb{T})\), Meisters and Schmidt showed that \(3\) differences are needed, in general. For the corresponding spaces on the real line, it is shown in [8] that \(2s + 1\) differences are needed, in general.

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References
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