

12 An Introduction to Fractals

The aim of this lecture is to give you a *brief* introduction to fractals, sufficient to be able to answer the assignment questions³. You may answer the assignment questions working in pairs and providing on set of answers containing both of your names.

In answering the assignment questions you will find it useful to explore various webpages devoted to fractals. You can find a list of starting places at

<http://www.ma.adfa.oz.au/~mark/teaching.dir/fractals.html>

To give you time to get acquainted with these, and associated, web sites instead of having a lecture on Friday we are providing you with a free period. We advise you to use this time wisely.

By the end of this section you should:

- Be able to give a non-technical explanation of what a fractal is.
- Explain how fractals can be generated by simple iterative processes.
- Understand some of the applications of fractals.
- Appreciate that fractals can arise in simple models of physical processes.

12.1 The Mandelbrot Set and Self-Similarity

One of the most intricate and beautiful images in mathematics is the Mandelbrot set, discovered by Benoit Mandelbrot in 1980. Mandelbrot's discovery and popularisation of this image is responsible for the development of fractal science.

Figure 16 (a) shows the *Mandelbrot Set*. I won't describe what the definition of the Mandelbrot Set is, that's assignment question 1! For our purposes it suffices to note that the Mandelbrot Set divides the plane into two regions: an 'inner region', the black region in the figure, and an 'outer region'. The boundary between the two regions is a *fractal*. What is a fractal? I'll discuss that latter.

Figures 16 (b-d) & 17 show successive magnifications of figure 16 (a). These images shows that additional detail on a finer scale is revealed at every magnification. Suppose that we can pick a point \mathcal{P} on the boundary. Points that are arbitrarily close to \mathcal{P} may be either in the 'inner' or 'outer' region. What does this mean? Suppose that these regions correspond to outcomes of an experiment and that the Mandelbrot set shows to which region a given initial condition (x, y) evolves to. Then we cannot predict the behaviour of a experiment when the initial condition is near to the boundary because arbitrarily small changes to the initial condition change the long-term dynamics of the system.

A figure is said to be *self-similar* if magnified subsets of the objects 'look like' the whole and to each other. The subsets need not look *exactly* the same as each other at all scales, but the same "type" of structures must appear on all scales. Not all self-similar objects are fractal, but all fractals are self-similar. Thus fractals have an infinite amount of detail.

Consider a smooth curve. Does it have the property of being self-similar? No. If we pick a point on the curve and magnify around it then the curve will eventually look like a line. It doesn't matter how complicated the curve is, it loses all of its detail as it is magnified. This *never happens* for a fractal. There is always more detail as we magnify it.

Now consider a sphere. Does it have the property of being self-similar? No. At significantly large magnification, it stops looking like a sphere and starts looking like a flat plane. For example, consider the earth. We know that it is a sphere, but if you look through a window you'll see no indication of this⁴ (unless you happen to be in orbit whilst reading this!).

³These are due in following the lecture on Wednesday 12th June.

⁴Hence the old belief in the Earth being flat.

Assignment Question 1 *What is the mathematical definition of the Mandelbrot Set?*

2 marks

12.1.1 What is a fractal?

Fractals are geometric figures, just like rectangles, circles and squares. However, we saw in the previous section that fractals have special properties that those figures do not have. What are these ‘special properties’ that differentiate fractals from non-fractals?

The scientific study of fractals is sufficiently new that there is not yet a widely accepted definition of what a fractal is. Different researchers have different ideas, consequently there are several competing definitions. This is contrast with the case for rectangles, circles and squares; there is no conflict of opinion over what a rectangle is! One definition of a fractal is:

A geometric figure or natural object is said to be fractal if it combines the following characteristics: (a) its parts have the same form or structure as the whole, except that they are at a different scale and may be slightly deformed; (b) its form is extremely irregular, or extremely interrupted or fragmented, and remains so, whatever the scale of examination; (c) it contains ‘distinct elements’ whose scales are very varied and cover a large range.” (B. Mandelbrot. *Les Objets Fractales* 1989, p.154)

Another characterisation of a fractal is:

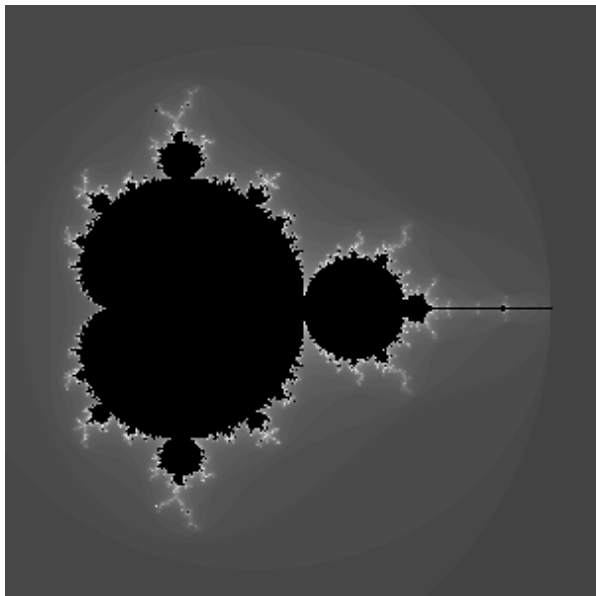
- (a) A complex structure at any level of magnification.
- (b) A non-integer dimension. We know that the dimension of lines, squares and cubes are respectively 1, 2 and 3. The dimension of a fractal may be 1.342.
- (c) They have a perimeter of infinite length but an area limited.
- (d) They are self-similar and independent of scale.

Assignment Question 2 *Who devised the word ‘fractal’?*

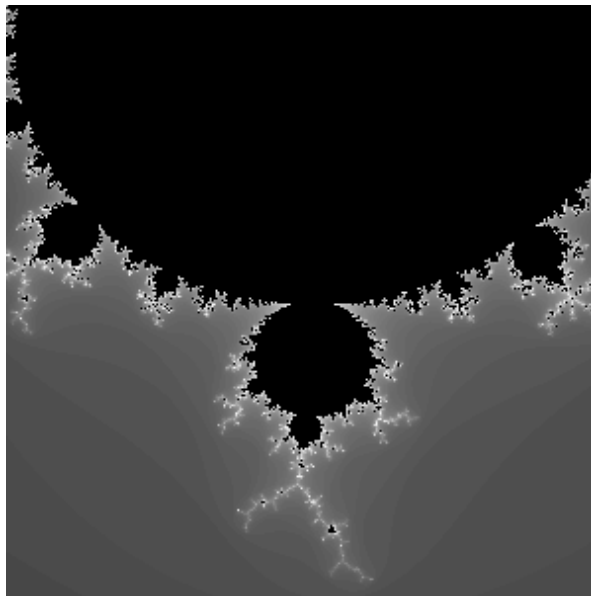
1 mark.

Assignment Question 3 *Describe one way to determine the dimension of an object.*

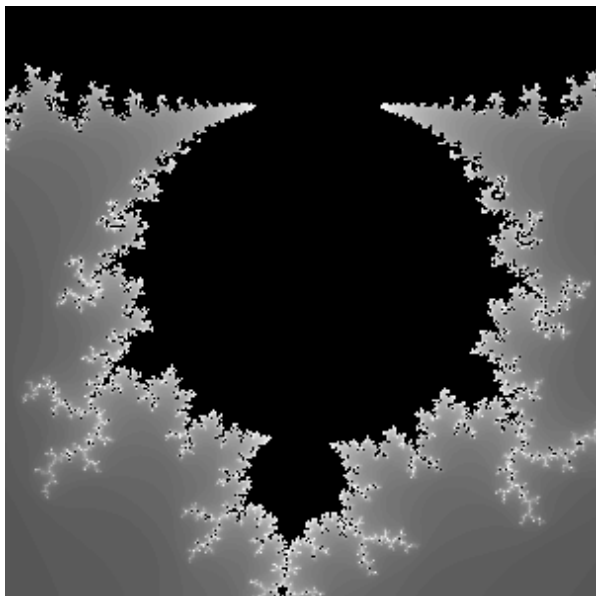
3 marks.



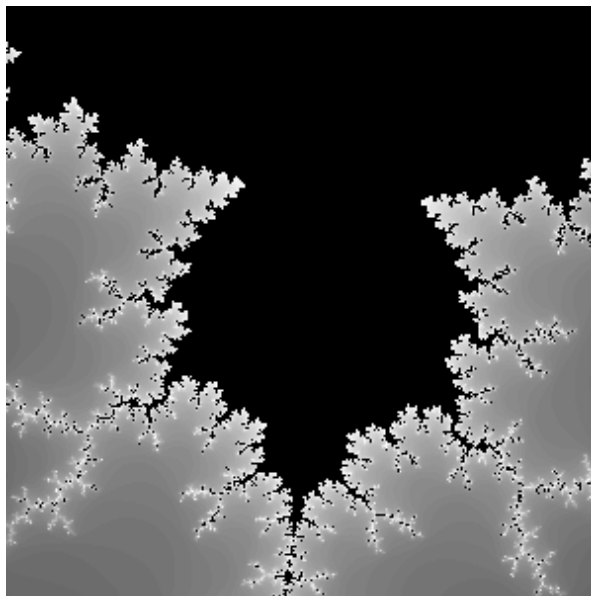
(a) The Mandelbrot Set



(b) A blow-up of figure (a)

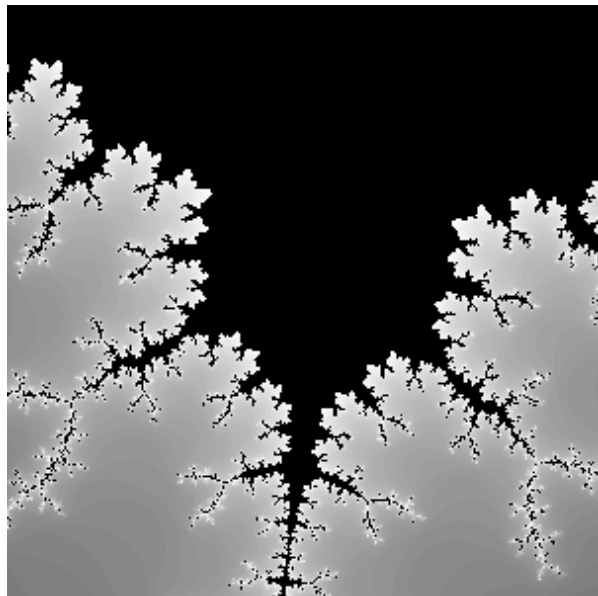


(c) A blow-up of figure (b).

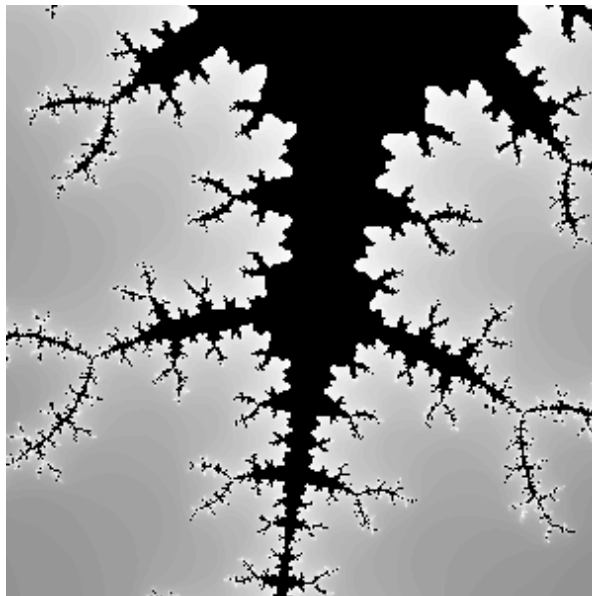


(d) A blow-up of figure (c)

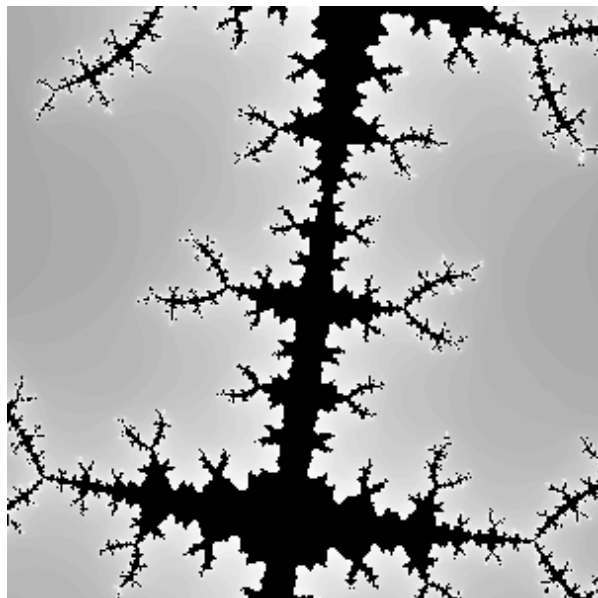
Figure 16: The Mandelbrot Set. Images obtained using the Julia and Mandelbrot Set Explorer at aleph0.clarku.edu/~djoyce/julia/explorer.html



(a) A blow-up of figure 16 (d)



(b) A blow-up of figure (a)



(c) A blow-up of figure (b).



(d) A blow-up of figure (c)

Figure 17: The Mandelbrot Set. Images obtained using the Julia and Mandelbrot Set Explorer at aleph0.clarku.edu/~djoyce/julia/explorer.html

12.2 Making a Fractal: the Sierpinski Triangle

In this section we learn how to make a famous fractal called the *Sierpinski Triangle*. You will find it useful to download some triangular grid paper via my web page to do this. The Sierpinski Triangle is named after the Polish mathematician Waclaw Sierpinski who discovered and investigated it in the 1920s.

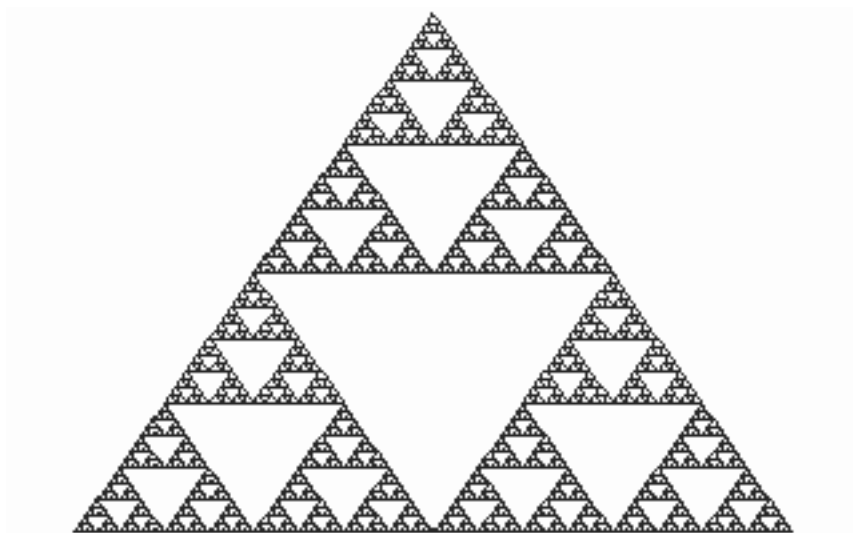


Figure 18: The fifth iteration of the Sierpinski Triangle.

Step One Draw an equilateral triangle with sides of 2 triangle lengths each. Connect the midpoints of each side.



Figure 19: The zeroth iterate of the Sierpinski Triangle.

How many equilateral triangles do you have now?

Shade out the triangle in the centre. Think of this as cutting a hole in the triangle.



Figure 20: The first iterate of the Sierpinski Triangle.

Step Two Draw another equilateral triangle with sides of 4 triangle lengths each. Connect the midpoints of the sides and shade the triangle in the centre as before.

Notice the three small triangles that also need to be shaded out in each of the three triangles on each corner - three more holes.



Figure 21: The second iterate of the Sierpinski Triangle.

Step Three Draw an equilateral triangle with sides of 8 triangle lengths each. Follow the same procedure as before, making sure to follow the shading pattern. You will have 1 large, 3 medium, and 9 small triangles shaded.



Figure 22: The third iterate of the Sierpinski Triangle.

Step Four If you follow the above pattern you should obtain

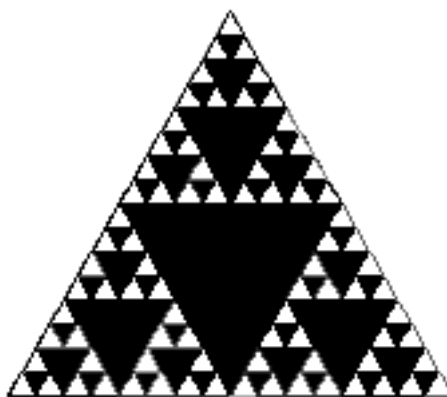


Figure 23: The fourth iterate of the Sierpinski Triangle.

Answer *either* assignment question (4) or assignment question (6).

Assignment Question 4 (Lanius)

1. Look at the triangle you made in Step One. What fraction of the triangle did you NOT shade?
2. What fraction of the triangle in Step Two is NOT shaded?
3. What fraction did you NOT shade in the Step Three triangle?
4. Do you see a pattern here? Use the pattern to predict the fraction of the triangle you would NOT shade in the Step Four Triangle. Confirm your prediction and explain.
5. CHALLENGE: Develop a formula so that you could calculate the fraction of the area which is NOT shaded for any step.

6. Write the fractions in the above questions in order from least to greatest. Write a statement about how their order connects to the shading out process.

10 marks.

12.2.1 Self-Similarity revisited

In section 12.1 we introduced the idea of self-similarity. When parts of some object are similar to the entire object, we call it *self-similar*.

In many fractals self-similarity is very obvious. For example, the Sierpinski triangle is composed of smaller versions of itself. When magnified, they turn out to be identical to the entire picture. This is known as *perfect self-similarity*. Sometimes the object is not perfectly self-similar. In such cases, the object is called *approximately self-similarity*.

12.2.2 Acknowledgements

Figure 18 was downloaded from <http://library.thinkquest.org/26242/full/>. The other figures, the description of the method used to obtain the Sierpinski Triangle and assignment question (4) came from <http://math.rice.edu/~lanius/fractals/>.

12.3 Making a Fractal: the Koch Snowflake

In this section we learn how to make a famous fractal called the *Koch Triangle*. You will find it useful to download some triangular grid paper via my web page to do this. The Koch Snowflake first appeared in a paper published by the Swedish mathematician Niels Fabian Helge von Koch in 1906.

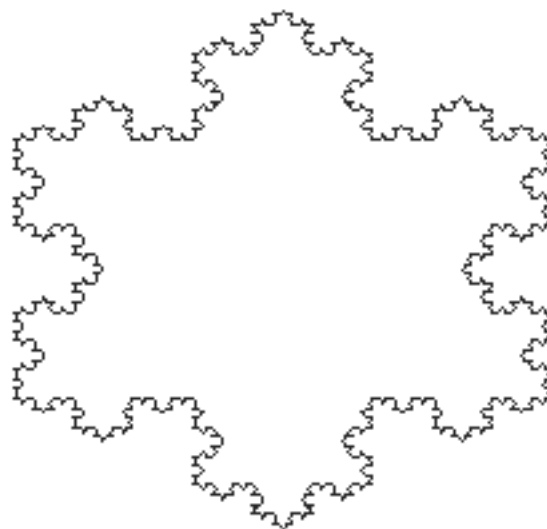


Figure 24: Approximation to the Koch snowflake after a large number of iterations.

Step One. Start with a large equilateral triangle. (See figure 25).

Step Two. Make a star.

1. Divide one side of the triangle into three parts and remove the middle section.
2. Replace it with two lines the same length as the section you removed.
3. Do this to all three sides of the triangle. (See figure 26 (a)).

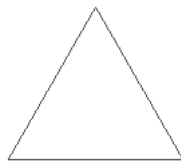


Figure 25: Zeroth iterate of the Koch Snowflake.

Do it again and again

The first four iterations of the Koch snowflake are shown in figure 26.

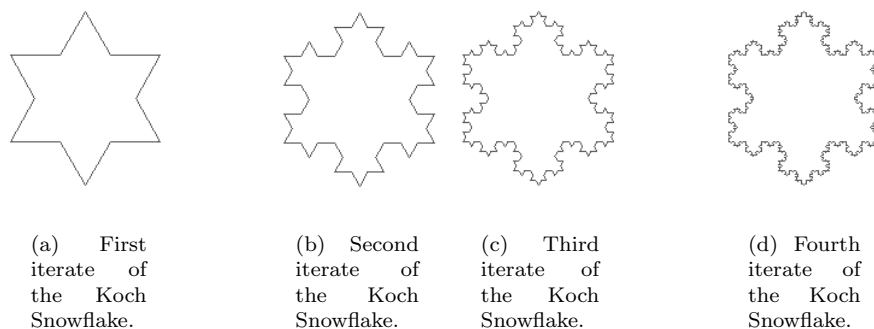


Figure 26: The first four iterations of the Koch Snowflake.

After an ‘infinite’ number of iterations the Koch snowflake looks something like figure 24.

Ordinarily, when you increase the perimeter of a geometric figure, you also increase its area. If you have a square with a huge parameter, it also has a huge area.

Assignment Question 5 *If the perimeter of a square is P what is its area?*

1 mark

Answer *either* assignment question (4) or assignment question (6).

Assignment Question 6 (Lanius)

1. *If the perimeter of the equilateral triangle you start with, i.e. figure 25, is 9 units what is the perimeter of figures 26 (a) & 26 (b).*
2. *Is there a pattern here? The perimeter of each figure is _____ times the perimeter of the figure before.*
3. *If the original triangle has a perimeter of 9 units, how many iterations would it take to obtain a perimeter of 100 units? (Or as close to 100 as you can get.)*
4. *Now think of doing this many, many times. The perimeter gets huge! But does the area? We say the area is bounded by a circle surrounding the original triangle. If you continued the process oh, let's say, infinitely many times, the figure would have an infinite perimeter, but its area would still be bounded by that circle.*

10 marks

If you are interested in how the area of the Koch Snowflake changes during the iterative process then you should read the following web page:

<http://math.rice.edu/~lanius/frac/koch3.html>

12.3.1 Acknowledgements

The description of the method to obtain the Koch Snowflake is taken from <http://math.rice.edu/~lanius/frac/koch.html>. Question (6) comes from <http://math.rice.edu/~lanius/frac/koch2.html>. Figure 24 was downloaded from <http://library.thinkquest.org/26242/full/>. The other figures were downloaded from <http://math.rice.edu/~lanius/frac/koch.html>.

12.4 Applications of Fractals

Clouds are not spheres, mountains are not cones, coastlines are not circles, and bark is not smooth, nor does lightning travel in a straight line. (B. Mandelbrot)

12.4.1 The use of fractals to create complex images

Many real-world objects can be thought of as being fractal-like: coastlines, clouds, mountains, turbulence, roots and branches of trees and veins and lungs of animals. These structures are fractal-like because they are approximately self-similar.

Consider the coast of Australia. From a great distance, such as from outer-space, the coastline looks like it is made up by segments of straight lines. As you approach each of these segments reveals additional detail in the form of bays, capes, harbours, and peninsulas that you could not previously see. From a sufficiently large distance it appears that each of these new features is made up by segments of straight lines. Picking a particular harbour and approaching it you will see that it too has additional detail in the form of smaller bays, capes, harbours, and peninsulas. At whatever scale you examine the coast-line it is *self-similar*; even if you are examining a grain of sand on the beach you will see thousands of indentations and extrusions. As we magnify the coastline ever greater detail is revealed.

Figure 27 shows a blow-up of the Koch Snowflake. No matter how much you magnify this fractal, you will see a fragmented picture. Compared to a straight line it is obviously better in describing a natural shape such as a coastline or a river. There are indentations, like harbours in a coastline; look closer and the harbour has bays, harbours, capes, and peninsulas. However, it has one major drawback: it has too much symmetry and regularity to be mistaken for a natural feature. In order to create more realistic images of natural objects a different type of self-similarity is used: *Brownian*, or *statistical self-similarity*. This process gives the fractal randomness, making them more realistic.

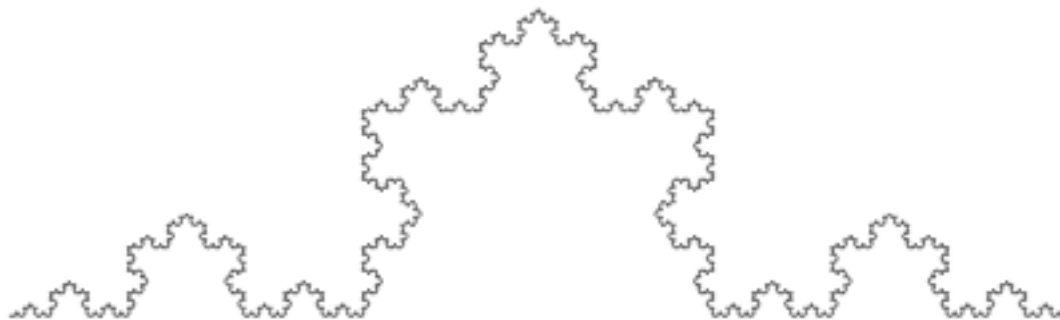


Figure 27: A blow-up of the Koch Snowflake.

Another example of a ‘natural fractal’ is a landscape. If you examine a mountain you will find that it is a complicated shape containing smaller hills and valleys. Looking at a hill, you will discover that it is composed of even smaller hills and valleys. On a particular hill you discover a rock. Breaking it you notice that its jagged edge has the same irregularities as a ridgeline on a horizon. The rock also looks like an entire mountain. As landscapes possess this fractal nature fractal algorithms can be used to generate terrain.

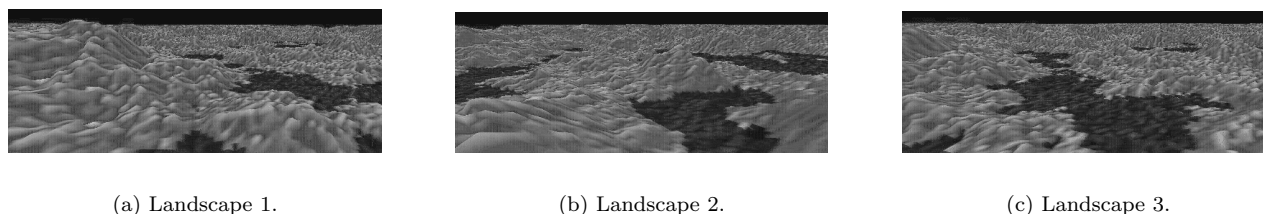


Figure 28: Three fractal landscapes.

In fact fractals can be used to generate images of any of the fractal-like real-world phenomena mentioned in the introduction to this section.

Assignment Question 7 *Outline some of the applications of generating landscape features using fractal algorithms.* 3 marks

12.4.2 How long is the coastline?

Suppose that we measure the length of the Australian coastline using a mile-long ruler, we would get a certain measurement. As a mile-long ruler is a coarse measure there would be many jagged features along the coastline that we would be unable to measure. Suppose instead that we used a foot-long ruler. We could now get into more nooks and crannies and we would record a higher length. Now, what about if we used an inch-long ruler? We could really get into the teeniest and tiniest of crannies. So the measurement would be even larger. Of course even with an inch long ruler there are nooks and crannies that we can see, but can't measure. As we use shorter and shorter rulers the length that we measure gets larger and larger. We reach the surprising conclusion that although the area of Australia is finite, the size of its coastline is infinite! The fact that the length of the coastline increases as the size of the ruler used decreases is known as the *coastline paradox*. This was first considered by L.F. Richardson.

12.4.3 Fractal Image Compression

The realization that a small, simple set of instructions can create a complex image has led to research in a new field known as *fractal image compression*. The idea is to store the simple, recursive instructions for creating the image rather than storing the image itself. For instance, a colour full-screen GIF image of the Mandelbrot Set occupies about 35 kilobytes. However, all you need to store it is the formula $z = z^2 + c$; when you need your image you simply iterate the formula and regenerate the image. As the formula takes no more than 7 bytes we have achieved a 99.98% compression!

The goal of fractal image compression is, given an image, to find functions each of which produces some part of the image. For a complex image that is not a fractal, you might need hundreds of such functions. However, the storage of hundreds of functions takes up considerable less space than hundreds of thousands of coloured pixels.

Assignment Question 8 *Write an essay about **one** of the following subjects. (No more than 2 A4 pages). Provide the URL of any web sites that you use.*

1. *Brownian, or statistical self-similarity.*
2. *The use of fractals to simulate terrain.*
3. *Fractal art.*
4. *Fractal music.*
5. *Fractal-image compression*

6. *The use of fractals to simulate plant forms.*

10 marks

12.4.4 Acknowledgements

Figure 27 was downloaded from <http://library.thinkquest.org/26242/full/>.

Figure 28 was downloaded from http://www.fciencias.unam.mx/Graf/fractales/fract_I.html.

12.5 Modelling

The Mandelbrot Set, section 12.1, the Sierpinski Triangle, section 12.2, and the Koch snowflake are all formed by *iterative* algorithms⁵. These algorithms are mathematical constructs, can fractals occur in models of physical systems? The answer is yes! Recall that when we discussed figures 16 & 17 we talked about the ‘inner’ and ‘outer’ regions. Here are some physical systems where the boundary between two types of behaviour is fractal.

Electrical power systems. Moving from the ‘inner’ to the ‘outer’ region represents power generators slipping out of synchronization resulting in the blackout of a city.

Electrical receivers. Here the ‘inner’ region represents a locked configuration and moving from the ‘inner’ to the ‘outer’ region results in a loss of communication.

Civil and aerospace engineering. A compressed structure is often required to operate in a metastable state. Moving from the ‘inner’ to the ‘outer’ region corresponds to collapse of the structure.

Naval architecture. Here we are interested in the stability of a structure in adverse weather conditions. Moving from the ‘inner’ to the ‘outer’ region corresponds to the capsizing of vessels or offshore oil production facilities.

In each of these systems it is important to establish the location of the fractal boundary so that we can identify if we are in a safe region, an unsafe region or a region in which we can not accurately determine if we are safe/unsafe.

When modelling physical systems we frequently think in terms of ordinary differential equations. The physical situations referred to above can often be modelled by the behaviour of a general mechanical oscillatory having one degree of freedom and with inertia, linear viscous damping, linear stiffness and a quadratic stiffness nonlinearity. After appropriate scaling this system can be written in the form

$$x'' + \beta x' + x - x^2 = F \sin \omega t, \quad (103)$$

where x is the dependent variable and a prime (') denotes differentiation with respect to the (scaled) time t . The positive coefficient β represents the magnitude of damping, and the system is driven by a sinusoidal force of magnitude F and frequency ω .

Fractal behaviour can therefore occur in very simple models of physical systems.

⁵Assignment question 1 asks you to define the iterative process used to obtain the Mandelbrot Set.