4.2 Polar Curves

4.2.1 Introduction

Polar curves are often (but not always) given in the form $r = f(\theta)$. For example

$$r = 6\sin\theta$$
$$r = 4(1 + \cos\theta)$$
$$r = 10.$$

When sketching curves, it is sometimes easier to convert the polar equation to a Cartesian equation, or vice versa, and then sketch the curve.

4.2.2 Converting polar equations to Cartesian equations (and vice versa)

Example

Convert the polar equation

$$r = \frac{-5}{\cos\theta}$$

to a Cartesian equation.

$$r = \frac{-5}{\cos\theta} \qquad \Rightarrow r\cos\theta = -5$$

$$\therefore \underline{x} = -5.$$

Example Convert the Cartesian equation

$$xy = 4$$

to a polar equation. Now, $x =$ ______ and $y =$ ______.

$$\therefore \quad xy = 4 \quad \text{becomes}$$

$$\underline{\hspace{4cm}} = 4$$

$$\text{i.e.} \quad \underline{\hspace{3.5cm}} = 4$$

$$\frac{}{\underline{\hspace{2.5cm}}} = 4$$

$$\therefore \quad r = \sqrt{\frac{8}{\sin 2\theta}}.$$

$$r\cos\theta$$

$$r\sin\theta$$

$$r\cos\theta\cdot r\sin\theta$$

$$r^2\cos\theta\sin\theta$$

$$\frac{r^2}{2}\sin 2\theta$$

Example Convert the Cartesian equation

$$x^2 + y^2 - 4x = 0$$

to a polar equation. Now $x =$ _____ and $y =$ _____.

$$\text{Thus, } x^2 + y^2 = \underline{\hspace{6cm}}$$
$$= \underline{\hspace{6cm}}$$
$$= \underline{\hspace{0.6cm}}$$

$\therefore$ we have

$$x^2 + y^2 = 4x$$
$$\underline{\hspace{0.6cm}} = \underline{\hspace{2.4cm}}$$
$$\underline{\hspace{4.5cm}} = 0$$

Thus $r = 0$ or $r - 4\cos\theta = 0$.

$r\cos\theta$

$r\sin\theta$

$r^2\cos^2\theta + r^2\sin^2\theta$

$r^2\left(\sin^2\theta + \cos^2\theta\right)$

r^2

r^2 $4r\cos\theta$

$r\left(r - 4\cos\theta\right)$

BUT, $r = 0$ is a point and it satisfies the equation $r - 4\cos\theta = 0$ when $\theta = \frac{\pi}{2}$. Therefore we can represent the curve as $r - 4\cos\theta = 0$ or $r = 4\cos\theta$.

Example Convert the polar equation

$$r^2 = 4\sin 2\theta$$

to a Cartesian equation. Recall that $\sin 2\theta =$ __________.

Thus

$$\begin{aligned} r^2 &= 4\sin 2\theta \\ &= 8\sin\theta\cos\theta \\ &= \\ & \underline{\hspace{6cm}} \\ \therefore\ \underline{\hspace{0.6cm}} &= \underline{\hspace{5cm}} \\ &= \underline{\hspace{5cm}} \\ \underline{\hspace{1.5cm}} & \\ \therefore\ \left(x^2 + y^2\right)^2 &= 8xy \end{aligned}$$

$$2\sin\theta\cos\theta$$

$$8\left(\frac{r\sin\theta}{r}\right)\left(\frac{r\cos\theta}{r}\right)$$

$$r^4 \qquad 8\,(r\sin\theta)\,(r\cos\theta)$$

$$\left(r^2\right)^2 \qquad 8\,(r\sin\theta)\,(r\cos\theta)$$

4.2.3 Sketching Curves

Example Sketch the following polar curves on the figures given below.

1. $r\cos\theta = -4$.
2. $r = 3$.

1. Converting $r\cos\theta = -4$ to the Cartesian equation $x = -4$ makes sketching easy.
2. Any choice of θ will give $r = 3$. Also note that

$$\begin{aligned} r = \sqrt{x^2 + y^2} &= 3 \\ \text{i.e.} \qquad x^2 + y^2 &= 9 \end{aligned}$$

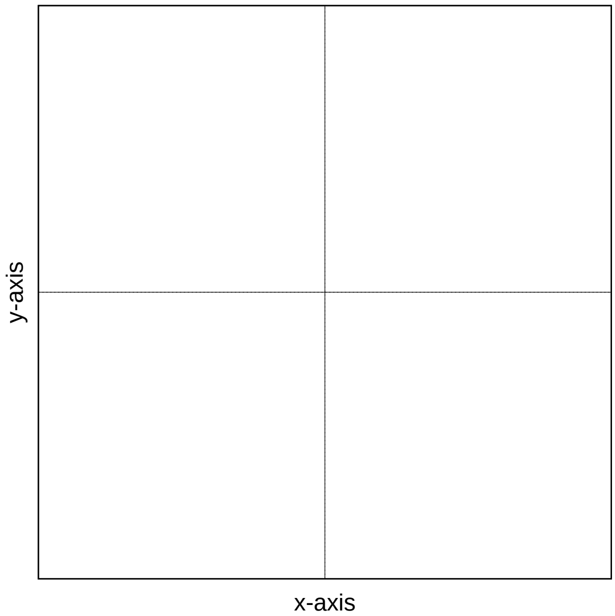


Figure 1: $r\cos\theta = -4$

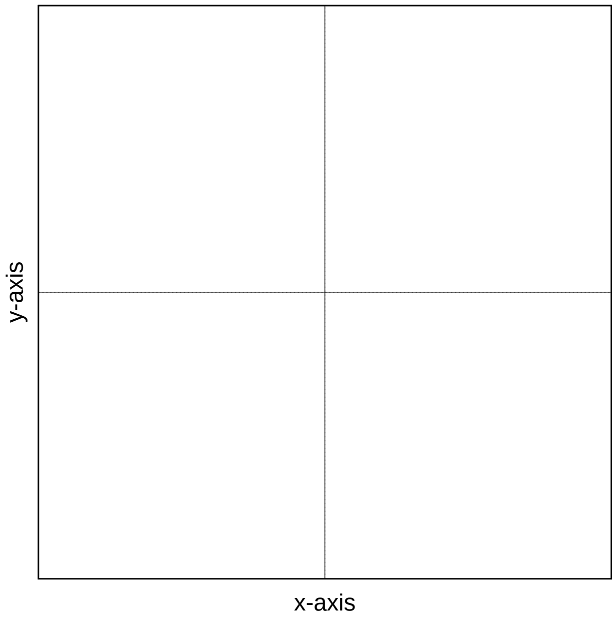


Figure 2: $r = 3$

Example Sketch the polar curve

$$r = 2 - 4\cos\theta$$

by drawing up a table of values.

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π
r	-2	-1.464	-0.828	0	2	4	4.828	5.464	6

θ	$\frac{7\pi}{6}$	$\frac{5\pi}{4}$	$\frac{4\pi}{3}$	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$	$\frac{7\pi}{4}$	$\frac{11\pi}{6}$	2π
r	5.464	4.828	4	2	0	-0.828	-1.464	-2

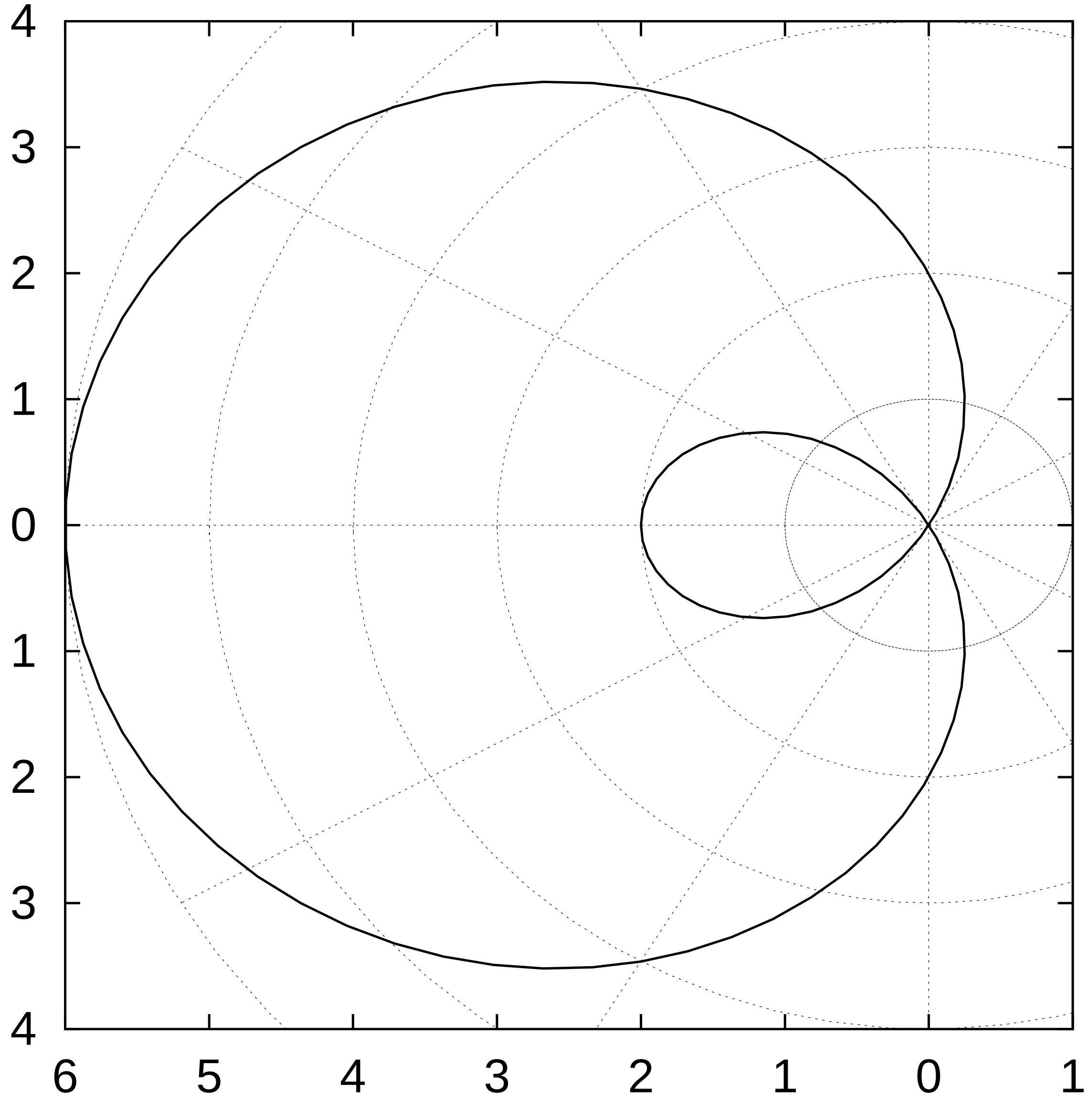


Figure 3: The graph $r = 2 - 4\cos\theta$.

1. Note that the rays are $\frac{\pi}{12}$ (or 15^o) apart.
2. Choose an appropriate scale for the concentric circles (don't make the figure too small!).
3. The values of θ in the table are the *minimum* required for a sketch.
4. You *should* attempt to draw the graph determined by the data in the table on polar graph paper.

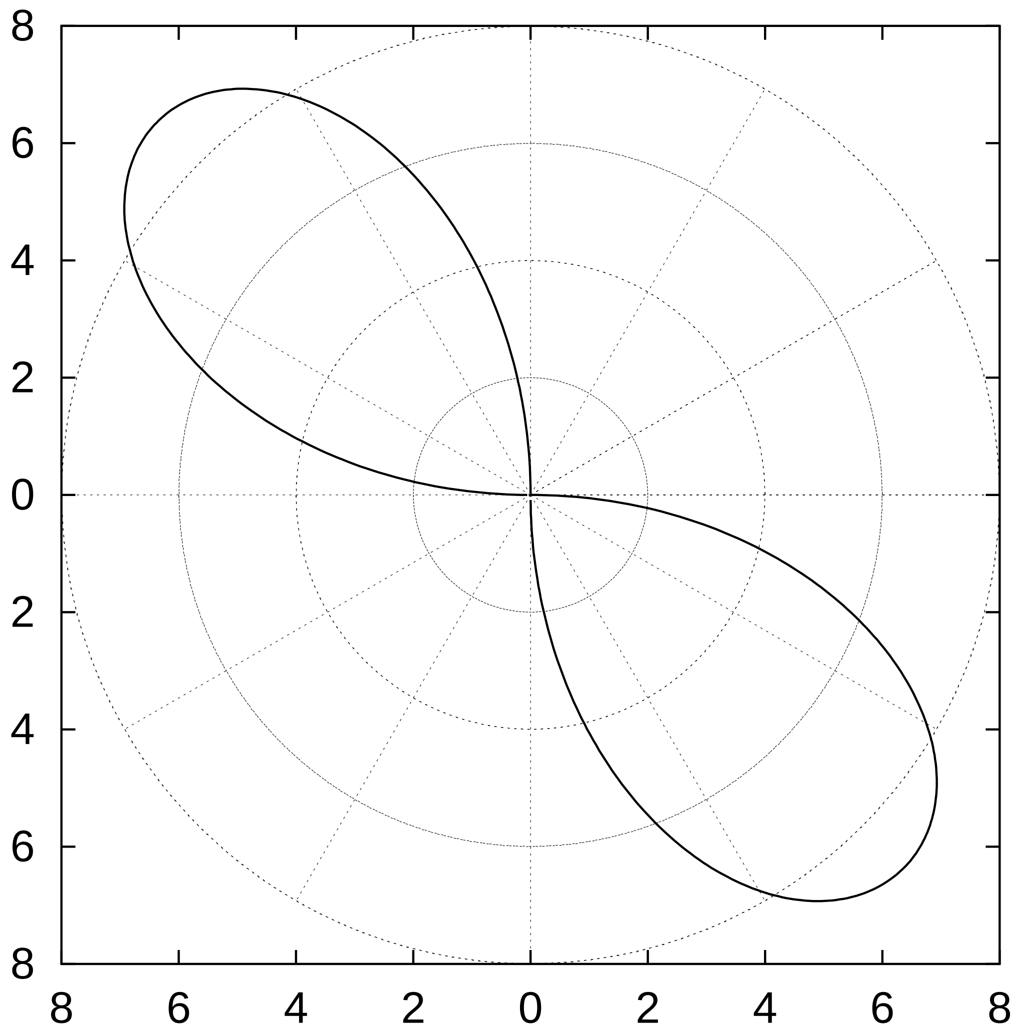
Note

1. If the table of values does not give a clear pattern, increase the selection of angles (in this case, consider $\frac{\pi}{12}, \frac{\pi}{8}, \ldots$)
2. Make sure the points are joined in the correct order.
3. Just as the points in a Cartesian graph (x and y) are joined by smooth curves, so too are the points joined when sketching polar curves.

Example The following example is left an exercise for you to do in your own time. Sketch the polar curve

$$r^2 = -9\sin 2\theta.$$

This curve is shown below.



4.2.4 Exercises