

3.3 Methods of integration

3.3.1 Integrals by inspection

Integration is the reverse of differentiation. Our standard derivatives provide us with a source of standard integrals that we can evaluate on sight.

$$\frac{d}{dx} \left(\frac{x^{n+1}}{n+1} \right) = x^n$$

$$\therefore \int x^n dx = \frac{x^{n+1}}{n+1} + c, \quad n \neq -1$$

$$\frac{d}{dx} (ax) = a$$

$$\therefore \int a dx = ax + c$$

$$\frac{d}{dx} (\sin x) = \cos x$$

$$\therefore \int \cos x dx = \sin x + c$$

$$\frac{d}{dx} (\cos x) = -\sin x$$

$$\therefore \int \sin x dx = -\cos x + c$$

$$\frac{d}{dx}(e^x) = e^x$$

$$\therefore \int e^x = e^x + c$$

$$\frac{d}{dx}(\ln x) = \frac{1}{x}$$

$$\therefore \int \frac{1}{x} dx = \ln x + c$$

$$\frac{d}{dx}(\sinh x) = \cosh x$$

$$\therefore \int \cosh x dx = \sinh x + c$$

$$\frac{d}{dx}(\cosh x) = \sinh x$$

$$\therefore \int \sinh x dx = \cosh x + c$$

Example Evaluate

$$1. \int (3x^6 - 2x^2 + 7x + 1) dx$$

$$2. \int \left(\frac{4}{\sqrt{x}} + \frac{1}{2x^2} \right) dx$$

$$3. \int_0^\pi (3e^x - 4 \cos x) dx$$

$$4. \int_1^2 \left(x^{-3} + \frac{1}{x} - \sinh x \right) dx$$

Answers

$$1. \frac{3x^7}{7} - \frac{2x^3}{3} + \frac{7x^2}{2} + x + c$$

$$2. 8\sqrt{x} - \frac{1}{2x} + c$$

$$3. 3(e^\pi - 1)$$

$$4. \frac{3}{8} + \ln 2 - \cosh 2 + \cosh 1$$

3.3.2 Simplifying Integrals

Sometimes an integrand can be simplified before evaluating the integral.

Examples

$$\int \frac{t^2 - 2t^4}{t^4} dt =$$

$$=$$

$$=$$

$$\int (2 + x^2)^2 dx =$$

$$=$$

$$\int \left(\frac{1}{t^2} - 2 \right) dt$$

$$\int (t^{-2} - 2) dt$$

$$-1t^{-1} - 2t + c$$

$$\int (4 + x^4 + 4x^2) dx$$
$$4x + \frac{x^5}{5} + \frac{4}{3}x^3 + c$$

$$\int_0^{\frac{\pi}{2}} \frac{\sin 2x}{\cos x} dx =$$

=

=

=

=

=

$$\int \frac{2 \sin x \cos x}{\cos x} dx$$

$$\int 2 \sin x dx$$

$$2 [-\cos x]_0^{\pi/2}$$

$$2 \left[-\cos \left(\frac{\pi}{2} \right) + \cos (0) \right]$$

$$2$$

Exercise Evaluate the following integrals.

1. $\int \frac{1}{\sec x} dx$

2. $\int_0^{\pi} (\cos^2 x + \sin^2 x) dx$

3. $\int \frac{x^2 \sin x + 2 \sin x}{x^2 + 2} dx$

4. $\int x^{1/3} (2 - x) dx$

5. $\int_0^1 \frac{x^2 - 1}{x + 1} dx$

6. $\int \sqrt[3]{x^2} dx$

$c =$ arbitrary constant

$$\sin(x) + c$$

π

$$-\cos(x) + c$$

$$\frac{3}{2}x^{4/3} - \frac{3}{7}x^{7/3} + c$$

$$-\frac{1}{2}$$

$$\frac{3}{5}x^{5/3} + c$$

3.3.3 Using Integral Tables

- The integration tables at the back of the course book contain 40 common integrals.
- The same tables will be included in the final exam for MATH 141.
- You should work out which integrals are in the tables and where they are.

Example Evaluate $\int \frac{dx}{x\sqrt{2x+4}}$

Solution Use formula [17].

$$\int \frac{1}{x\sqrt{ax+b}} dx = \frac{1}{\sqrt{b}} \ln \left| \frac{\sqrt{ax+b} - \sqrt{b}}{\sqrt{ax+b} + \sqrt{b}} \right| + c$$

In our question

$$a = \underline{\quad} \quad \quad \quad 2$$

$$b = \underline{\quad} \quad \quad \quad 4$$

Therefore we have

$$\int \frac{1}{x\sqrt{ax+b}} dx =$$

$$\frac{1}{\sqrt{4}} \ln \left| \frac{\sqrt{2x+4} - \sqrt{4}}{\sqrt{2x+4} + \sqrt{4}} \right|$$

Example Evaluate $\int \sin^5 x dx$

Solution Use formula [33]

$$\int \sin^n x dx = \frac{-1}{n} \cos x \sin^{n-1} x + \frac{n-1}{n} \int \sin^{n-2} x dx$$

$$n = \underline{\quad}$$

$$\int \sin^5 x dx = \frac{-1}{5} \cos x \sin^4 x + \frac{4}{5} \int \sin^3 x dx$$

We need to use the formula again to find $\int \sin^3 x dx$

$$\int \sin^n x dx = \frac{-1}{n} \cos x \sin^{n-1} x + \frac{n-1}{n} \int \sin^{n-2} x dx$$

$$n = \underline{\quad}$$

$$\begin{aligned} \int \sin^3 x dx &= \frac{-1}{3} \cos x \sin^2 x + \frac{2}{3} \int \sin x dx + c \\ &= \frac{-1}{3} \cos x \sin^2 x - \frac{2}{3} \cos x + c \end{aligned}$$

$$\begin{aligned} \int \sin^5 x &= \frac{-1}{5} \cos x \sin^4 x \\ &\quad - \frac{4}{15} \cos x \sin^2 x - \frac{8}{15} \cos x + c \end{aligned}$$

Example Evaluate $\int_{2e}^{3e} \frac{1}{e^2 - x^2} dx$

Solution Use formula [20].

$$\int \frac{1}{a^2 - x^2} dx = \frac{1}{2a} \ln \left| \frac{x+a}{x-a} \right| + c$$

$$a = \underline{\quad}$$

$$\int \frac{1}{e^2 - x^2} dx =$$

$$\int_{2e}^{3e} \frac{1}{e^2 - x^2} dx =$$

e

$$\frac{1}{2e} \ln \left| \frac{x+e}{x-e} \right| + c$$
$$\frac{1}{2e} \left[\ln \left| \frac{x+e}{x-e} \right| \right]_{2e}^{3e}$$

$$\begin{aligned}
&= \frac{1}{2e} \left[\ln \left(\frac{4e}{2e} \right) - \ln \left(\frac{3e}{e} \right) \right] \\
&= \frac{1}{2e} [\ln 2 - \ln 3] \\
&= \frac{1}{2e} \ln \left(\frac{2}{3} \right)
\end{aligned}$$

Example Evaluate $\int \frac{4x}{\sqrt{2x+7}} dx$

Solution Use formula [16].

$$\int \frac{x}{\sqrt{ax+b}} dx = \frac{2ax-4b}{3a^2} \sqrt{ax+b} + c$$

We need to re-write our integral so that we can use [16]

$$\int \frac{4x}{\sqrt{2x+7}} dx = 4 \int \frac{x}{\sqrt{2x+7}} dx$$

$$a = 2$$

$$b = 7$$

$$\int \frac{4x}{\sqrt{2x+7}} dx =$$

2

7

$$4 \left[\frac{2(2)x - 4(7)}{3(2)^2} \right] \sqrt{2x+7} + c$$

Exercise Use the 'Table of Integrals' to evaluate the following.

(a) $\int e^{2x} \sin 3x dx$

(b) $\int x\sqrt{9x+1} dx$

(c) $\int x(x+2)^4 dx$

(d) $\int_0^1 \frac{1}{1+3e^{2x}} dx$

(e) $\int xe^x dx$

(f) $\int_0^{\pi/2} x \sin x dx$

Exercise 3.3.4

$c =$ arbitrary constants

$$\frac{1}{13}e^{2x} (2 \sin 3x - 3 \cos 3x) + c$$

$$\frac{2}{81} \left[\frac{(9x+1)^{5/2}}{5} - \frac{(9x+1)^{3/2}}{3} \right] + c$$

$$(x+2)^5 \left(\frac{x+2}{6} - \frac{2}{5} \right) + c$$

$$\frac{1}{2} [2x - \ln(1 + 3e^{2x})] + c$$

$$xe^x - e^x + c$$