

# Integration

## 3.1 The indefinite integral and the definite integral

### 3.1.1 The indefinite integral

Suppose that

$$\frac{dy}{dx} = 1$$

**True or False** The function  $y$  is given by

$$y = x?$$

Differentiate the following functions

$$y = x \quad \frac{dy}{dx} = \underline{\hspace{2cm}}$$

$$y = x + 1 \quad \frac{dy}{dx} = \underline{\hspace{2cm}}$$

$$y = x + 2 \quad \frac{dy}{dx} = \underline{\hspace{2cm}}$$

$$y = x + 3 \quad \frac{dy}{dx} = \underline{\hspace{2cm}}$$

Suppose that

$$\frac{dy}{dx} = 1$$

Then

$$y = \underline{\hspace{2cm}}$$

We know that

$$\frac{d}{dx}(x^2) = \underline{\hspace{2cm}}$$

**True or False**

Therefore, the integral of  $2x$  with respect to  $x$  is  $x^2$ ?

This is written

$$\int 2x dx = x^2$$

where the symbols  $\int \dots dx$  denote 'the integral of  $\dots$  with respect to  $x$ '.

### 3.1.2 The antiderivative

#### Definition

Suppose that  $\frac{d}{dx}\mathcal{F}(x) = f(x)$ . The  $\mathcal{F}$  is called an antiderivative or indefinite integral or primitive of a function  $f$ .

The antiderivative  $\mathcal{F}$  is usually denoted by

$$\mathcal{F}(x) = \int f(x) dx$$

More generally,

$$\int f(x) dx = \mathcal{F}(x) + c$$

where  $c$  is known as the constant of integration.

Evaluate the integrals

1.  $\int 1 \cdot dx$

2.  $\int 2x dx$

3.  $\int 3x^2 dx$

4.  $\int \cos(x) dx$

5.  $\int \sec^2(x) dx$

6.  $\int \frac{1}{x} dx$

7.  $\int e^x dx$

8.  $\int \sin(x) dx$

$c = \text{constant}$

$$x + c$$

$$x^2 + c$$

$$x^3 + c$$

$$\sin x + c$$

$$\tan x + c$$

$$\ln x + c$$

$$e^x + c$$

$$-\cos x + c$$

Suppose that

$$\frac{d}{dx}\mathcal{F}(x) = f(x)$$

Hence

$$\int f(x)dx = \mathcal{F} + c$$

Thus

$$\begin{aligned}\frac{d}{dx}\mathcal{F}(x) &= \frac{d}{dx} \left[ \int f(x)dx \right] \\ &= f(x)\end{aligned}$$

$$\text{i.e. } \frac{d}{dx} \left[ \int f(x) dx \right] = f(x)$$

## 3.2 The Definite Integral

### 3.2.1 Fixed end points

If  $\mathcal{F}$  is the antiderivative of a function  $f$ , then the definite integral of  $f$  is given by

$$\int_a^b f(x)dx = [\mathcal{F}(x)]_a^b = \mathcal{F}(b) - \mathcal{F}(a)$$

**Example** Calculate

$$\int_0^2 1 \cdot dx = [x]_0^2 = \underline{\hspace{2cm}}$$

a and b are called the limits of integration, and x the dummy variable of integration. The function  $f(x)$  is called the integrand.

$$x \quad 2 - 0 = 2$$

**Question.** What does  $\int_a^b f(x) dx$   
'mean'?

Evaluate the definite integrals

1.  $\int_0^5 1 \cdot dx$

2.  $\int_1^5 x dx$

3.  $\int_2^3 6x^2 dx$

4.  $\int_0^{\pi/2} \cos(2x) dx$

5.  $\int_0^{\pi/4} \sec^2(x) dx$

6.  $\int_e^{e^2} \frac{4}{x} dx$

7.  $\int_0^2 5e^x dx$

5

12

38

0

1

4

$$5(e^2 - 1)$$

The value of the integral depends on the function to be integrated, not on the particular variable used, i.e.

$$\begin{aligned}\int_a^b f(x)dx &= \int_a^b f(t)dt = \int_a^b f(u)du \\ &= \dots = \mathcal{F}(b) - \mathcal{F}(a)\end{aligned}$$

e.g.

$$\begin{aligned}\int_1^2 2x dx &= \int_1^2 2t dt = \int_1^2 2u du \\ &= \left[ \underline{\hspace{1cm}} \right]_1^2 = \underline{\hspace{1cm}}\end{aligned}$$

$$x^2 \quad 4 - 1 = 3$$

### 3.2.2 Variable Endpoints

A definite integral can take the following form  $\int_a^x f(x)dx$  where the upper limit is allowed to vary. For such integrals it is best to use a letter different from  $x$  for the variable of integration; thus we write

$$\int_a^x f(t)dt \quad \text{rather than} \quad \int_a^x f(x)dx.$$

Both endpoints may be functions, for example

$$\begin{aligned} \int_{g(x)}^{h(x)} f(t)dt &= [\mathcal{F}(t)]_{g(x)}^{h(x)} \\ &= \mathcal{F}[h(x)] - \mathcal{F}[g(x)] \end{aligned}$$

Motivation?



Evaluate the integrals

$$1. \int_t^{t^2} 5 \cdot dx$$

$$5t(t-1)$$

$$2. \int_{\cos t}^{\sin t} \frac{x}{2} dx$$

$$\frac{1}{4} (\sin^2 t - \cos^2 t)$$

$$3. \int_0^t \cos\left(\frac{x}{4}\right) dx$$

$$4 \sin\left(\frac{t}{4}\right)$$

$$4. \int_{e^t}^{e^{2t}} \frac{1}{2x} dx$$

$$\frac{t}{2}$$

$$5. \int_t^{5t} \sin(x) dx$$

$$\cos(t) - \cos(5t)$$

We know that

$$\int_a^x f(t) dt = \mathcal{F}(x) - \mathcal{F}(a)$$

**Claim**

$$\frac{d}{dx} \int_a^x f(t) dt = f(x)$$

**Proof**

$$\begin{aligned} \frac{d}{dx} \int_a^x f(t) dt &= \frac{d}{dx} [\mathcal{F}(x) - \mathcal{F}(a)] \\ &= \frac{d}{dx} [\mathcal{F}(x)] - \frac{d}{dx} [\mathcal{F}(a)] \\ &= \frac{d}{dx} [\mathcal{F}(x)] \\ &= f(x). \end{aligned}$$

Motivation?

A more general result can be found using the chain rule.

**Claim**

$$\frac{d}{dx} \int_a^{g(x)} f(t) dt = f[g(x)] \cdot g'(x)$$

Motivation?

**Example** Evaluate  $\frac{d}{dx} \left[ \int_{\frac{\pi}{4}}^{x^2} \cos t dt \right]$  by

1. Using the above formula
2. By first integrating and then differentiating

1. Use the formula

$$a =$$

$$f = \underline{\quad}$$

$$g = \underline{\quad}$$

$$g' = \underline{\quad}$$

$$\therefore \frac{d}{dx} \left[ \int_{\frac{\pi}{4}}^{x^2} \cos t dt \right] = \underline{\hspace{2cm}}$$

$$\frac{\pi}{4}$$

$$\cos t$$

$$x^2$$

$$2x$$

$$\cos(x^2) \cdot 2x$$

2. (i) First do the integration

$$\int_{\frac{\pi}{4}}^{x^2} \cos t \, dt = \left[ \underline{\hspace{2cm}} \right]_{\frac{\pi}{4}}^{x^2}$$
$$= \underline{\hspace{10cm}}$$

(ii) Now do the differentiation

$$\frac{d}{dx} \left[ \int_{\frac{\pi}{4}}^{x^2} \cos t \, dt \right] = \underline{\hspace{10cm}}$$
$$= \underline{\hspace{10cm}}$$

$$\sin t$$

$$\sin (x^2) - \sin \left( \frac{\pi}{4} \right)$$

$$\frac{d}{dx} \sin (x^2) - \frac{d}{dx} \sin \left( \frac{\pi}{4} \right)$$

$$2x \cos (x^2)$$



$$(x^2 - 2x + 4) \cdot 1$$

**Example** Evaluate

$$\frac{d}{dx} \left[ \int_a^{x^2} (t^2 - 2t + 4)^{3/4} dt \right]$$

$$a = \underline{\quad}$$

$$f = \underline{\hspace{4cm}}$$

$$g = \underline{\quad}$$

$$g' = \underline{\quad}$$

$$\therefore \frac{d}{dx} \left[ \int_a^{x^2} (t^2 - 2t + 4)^{3/4} dt \right]$$

$$= \underline{\hspace{4cm}}$$

$$= \underline{\hspace{4cm}}$$

$a$

$$(t^2 - 2t + 4)^{3/4}$$

$$x^2$$

$$2x$$

$$\left[ (x^2)^2 - 2(x^2) + 4 \right]^{3/4} \cdot 2x$$
$$(x^4 - 2x^2 + 4)^{3/4} \cdot 2x$$



### 3.2.3 Properties of Integrals

$$(i) \int_a^b (f+g)(x)dx = \int_a^b f(x)dx + \int_a^b g(x)dx$$

$$(ii) \int_a^b (\alpha f)(x)dx = \int_a^b \alpha f(x)dx = \alpha \int_a^b f(x)dx$$

$$(iii) \int_a^a f(x)dx = 0$$

$$(iv) \int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$$

$$(v) \int_a^b f(x)dx = - \int_b^a f(x)dx$$

(vi) If  $f(x)$  is an *odd* function (i.e.,  
 $f(x) = -f(-x)$  for all  $x$ )  
 $\int_{-a}^a f(x)dx = 0.$

(vii) If  $f(x)$  is an *even* function (i.e.,  
 $f(x) = f(-x)$  for all  $x$ )  
 $\int_{-a}^a f(x)dx = 2 \int_0^a f(x)dx.$

(viii) If  $f(x) \leq g(x)$  for  $a \leq x \leq b$ , then  
 $\int_a^b f(x)dx \leq \int_a^b g(x)dx$

### 3.2.4 Exercises