Integration

3.1 The indefinite integral and the definite integral

3.1.1 The indefinite integral

Suppose that

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 1$$

True or False The function y is given by

$$y = x?$$

Differentiate the following functions

$$y = x \qquad \frac{\mathrm{d}y}{\mathrm{d}x} = \_\_\_\_\_$$

$$y = x + 1 \qquad \frac{\mathrm{d}y}{\mathrm{d}x} = \_\_\_\_\_$$

$$y = x + 2 \qquad \frac{\mathrm{d}y}{\mathrm{d}x} = \_\_\_\_\_$$

$$y = x + 3 \qquad \frac{\mathrm{d}y}{\mathrm{d}x} = \_\_\_\_\_$$

Suppose that

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 1$$

Then

$$y = \_\_\_\_\_\_\_\_\_\_$$

We know that

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(x^2\right) = \_\_\_\_\_$$

True or False

Therefore, the integral of $2x$ with respect to x is x^2?

This is written

$$\int 2x\mathrm{d}x = x^2$$

where the symbols $\int \ldots \mathrm{d}x$ denote 'the integral of . . . with respect to x'.

3.1.2 The antiderivative

Definition

Suppose that $\frac{\mathrm{d}}{\mathrm{d}x}\mathcal{F}(x) = f(x)$. The $\mathcal{F}$ is called an antiderivative or indefinite integral or primitive of a function f.

The antiderivative $\mathcal{F}$ is usually denoted by

$$\mathcal{F}(x) = \int f(x)\mathrm{d}x$$

More generally,

$$\int f(x)\mathrm{d}x = \mathcal{F}(x) + c$$

where c is known as the constant of integration.

Evaluate the integrals

1. $\int 1 \cdot \mathrm{d}x$

2. $\int 2x\mathrm{d}x$

3. $\int 3x^2\mathrm{d}x$

4. $\int \cos(x)\,\mathrm{d}x$

5. $\int \sec^2(x)\,\mathrm{d}x$

6. $\int \frac{1}{x}\mathrm{d}x$

7. $\int e^x\mathrm{d}x$

8. $\int \sin(x)\,\mathrm{d}x$

c=constant

$$x + c$$

$$x^2 + c$$

$$x^3 + c$$

$$\sin x + c$$

$$\tan x + c$$

$$\ln x + c$$

$$e^x + c$$

$$-\cos x + c$$

Suppose that

$$\frac{\mathrm{d}}{\mathrm{d}x}\mathcal{F}(x) = f(x)$$

Hence

$$\int f(x)\mathrm{d}x = \mathcal{F} + c$$

Thus

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}\mathcal{F}(x) &= \frac{\mathrm{d}}{\mathrm{d}x}\left[\int f(x)\mathrm{d}x\right] \\ &= f(x)\end{aligned}$$

$$\text{i.e.} \quad \frac{\mathrm{d}}{\mathrm{d}x}\left[\int f(x)\,\mathrm{d}x\right] = f(x)$$

3.2 The Definite Integral

3.2.1 Fixed end points

If $\mathcal{F}$ is the antiderivative of a function f, then the definite integral of f is given by

$$\int_a^b f(x)\mathrm{d}x = [\mathcal{F}(x)]_a^b = \mathcal{F}(b) - \mathcal{F}(a)$$

Example Calculate

$$\int_0^2 1 \cdot \mathrm{d}x = [\_]_0^2 = \underline{\qquad\qquad}$$

a and b are called the limits of integration, and x the dummy variable of integration. The function $f(x)$ is called the integrand.

$$x \qquad 2 - 0 = 0$$

Question. What does $\int_a^b f(x)\,\mathrm{d}x$ 'mean'?

Evaluate the definite integrals

1. $\int_0^5 1 \cdot \mathrm{d}x$

2. $\int_1^5 x \mathrm{d}x$

3. $\int_2^3 6x^2 \mathrm{d}x$

4. $\int_0^{\pi/2} \cos(2x)\, \mathrm{d}x$

5. $\int_0^{\pi/4} \sec^2(x)\, \mathrm{d}x$

6. $\int_e^{e^2} \frac{4}{x} \mathrm{d}x$

7. $\int_0^2 5e^x \mathrm{d}x$

$$5$$

$$12$$

$$38$$

$$0$$

$$1$$

$$4$$

$$5\left(e^{2}-1\right)$$

The value of the integral depends on the function to be integrated, not on the particular variable used, i.e.

$$\int_a^b f(x)\mathrm{d}x = \int_a^b f(t)\mathrm{d}t = \int_a^b f(u)\mathrm{d}u$$
$$= \ldots = \mathcal{F}(b) - \mathcal{F}(a)$$

e.g.

$$\int_1^2 2x\mathrm{d}x = \int_1^2 2t\mathrm{d}t = \int_1^2 2u\mathrm{d}u$$
$$= \big[\_\_\big]_1^2 = \_\_\_\_\_\_\_\_$$

x^2 $4 - 1 = 3$

3.2.2 Variable Endpoints

A definite integral can take the following form $\underline{\int_a^x f(x)\mathrm{d}x}$ where the upper limit is allowed to vary. For such integrals it is best to use a letter different from x for the variable of integration; thus we write

$$\int_a^x f(t)\mathrm{d}t \quad \text{rather than} \quad \int_a^x f(x)\mathrm{d}x.$$

Both endpoints may be functions, for example

$$\begin{aligned}\int_{g(x)}^{h(x)} f(t)\mathrm{d}t &= [\mathcal{F}(t)]_{g(x)}^{h(x)} \\ &= \mathcal{F}\left[h(x)\right] - \mathcal{F}\left[g(x)\right]\end{aligned}$$

Motivation?

Evaluate the integrals

1. $\int_{t}^{t^2} 5 \cdot \mathrm{d}x$

2. $\int_{\cos t}^{\sin t} \frac{x}{2} \mathrm{d}x$

3. $\int_{0}^{t} \cos\left(\frac{x}{4}\right) \mathrm{d}x$

4. $\int_{e^t}^{e^{2t}} \frac{1}{2x} \mathrm{d}x$

5. $\int_{t}^{5t} \sin(x) \, \mathrm{d}x$

$$5t\left(t-1\right)$$

$$\frac{1}{4}\left(\sin^{2}t-\cos^{2}t\right)$$

$$4\sin\left(\frac{t}{4}\right)$$

$$\frac{t}{2}$$

$$\cos\left(t\right)-\cos\left(5t\right)$$

We know that

$$\int_a^x f(t)\,\mathrm{d}t = \mathcal{F}(x) - \mathcal{F}(a)$$

Claim

$$\frac{\mathrm{d}}{\mathrm{d}x}\int_a^x f(t)\,\mathrm{d}t = f(x)$$

Proof

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}\int_a^x f(t)\,\mathrm{d}t &= \frac{\mathrm{d}}{\mathrm{d}x}\left[\mathcal{F}(x) - \mathcal{F}(a)\right]\\ &= \frac{\mathrm{d}}{\mathrm{d}x}\left[\mathcal{F}(x)\right] - \frac{\mathrm{d}}{\mathrm{d}x}\left[\mathcal{F}(a)\right]\\ &= \frac{\mathrm{d}}{\mathrm{d}x}\left[\mathcal{F}(x)\right]\\ &= f(x).\end{aligned}$$

Motivation?

A more general result can be found using the chain rule.

Claim

$$\frac{\mathrm{d}}{\mathrm{d}x}\int_a^{g(x)} f(t)\,\mathrm{d}t = f[g(x)]\cdot g'(x)$$

Motivation?

Example Evaluate $\frac{\mathrm{d}}{\mathrm{d}x}\left[\int_{\frac{\pi}{4}}^{x^2}\cos t\mathrm{d}t\right]$ by

1. Using the above formula
2. By first integrating and then differentiating

1. Use the formula

$$a = $$

$$f = \frac{}{}$$

$$g = \underline{}$$

$$g' = \underline{}$$

$$\therefore \frac{\mathrm{d}}{\mathrm{d}x}\left[\int_{\frac{\pi}{4}}^{x^2} \cos t \mathrm{d}t\right] = \underline{}$$

$$\frac{\pi}{4}$$

$$\cos t$$

$$x^2$$

$$2x$$

$$\cos\left(x^2\right) \cdot 2x$$

2. (i) First do the integration

$$\int_{\frac{\pi}{4}}^{x^2} \cos t \, \mathrm{d}t = [\_\_\_\_\_]_{\frac{\pi}{4}}^{x^2}$$

$$= \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_$$

(ii) Now do the differentiation

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[\int_{\frac{\pi}{4}}^{x^2} \cos t \mathrm{d}t\right] = \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_$$

$$= \_\_\_\_\_\_\_\_\_\_$$

$$\sin t$$

$$\sin\left(x^{2}\right) - \sin\left(\frac{\pi}{4}\right)$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\sin\left(x^{2}\right) - \frac{\mathrm{d}}{\mathrm{d}x}\sin\left(\frac{\pi}{4}\right)$$

$$2x\cos\left(x^{2}\right)$$

Example Evaluate

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[\int_0^x \left(t^2 - 2t + 4\right)\mathrm{d}t\right].$$

$$a = \_$$

$$f = \_\_\_\_\_\_\_\_\_\_$$

$$g = \_$$

$$g' = \_$$

$$\therefore \frac{\mathrm{d}}{\mathrm{d}x}\left[\int_0^x \left(t^2 - 2t + 4\right)\mathrm{d}t\right] =$$

____

x

$t^2 - 2t + 4$

x

1

$$\left(x^2 - 2x + 4\right) \cdot 1$$

Example Evaluate

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[\int_a^{x^2} \left(t^2 - 2t + 4\right)^{3/4} \mathrm{d}t\right]$$

$$a = \_$$

$$f = \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_$$

$$g = \_\_$$

$$g' = \_\_$$

$$\therefore \frac{\mathrm{d}}{\mathrm{d}x}\left[\int_a^{x^2} \left(t^2 - 2t + 4\right)^{3/4} \mathrm{d}t\right]$$

$$= \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_$$

$$= \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_$$

$$a$$

$$\left(t^2 - 2t + 4\right)^{3/4}$$

$$x^2$$

$$2x$$

$$\left[\left(x^2\right)^2 - 2\left(x^2\right) + 4\right]^{3/4} \cdot 2x$$

$$\left(x^4 - 2x^2 + 4\right)^{3/4} \cdot 2x$$

3.2.3 Properties of Integrals

(i) $\int_a^b (f+g)(x)\mathrm{d}x = \int_a^b f(x)\mathrm{d}x + \int_a^b g(x)\mathrm{d}x$

(ii) $\int_a^b (\alpha f)(x)\mathrm{d}x = \int_a^b \alpha f(x)\mathrm{d}x = \alpha \int_a^b f(x)\mathrm{d}x$

(iii) $\int_a^a f(x)\mathrm{d}x = 0$

(iv) $\int_a^b f(x)\mathrm{d}x = \int_a^c f(x)\mathrm{d}x + \int_c^b f(x)\mathrm{d}x$

(v) $\int_a^b f(x)\mathrm{d}x = -\int_b^a f(x)\mathrm{d}x$

(vi) If $f(x)$ is an *odd* function (i.e., $f(x) = -f(-x)$ for all x) $\int_{-a}^a f(x)\mathrm{d}x = 0.$

(vii) If $f(x)$ is an *even* function (i.e., $f(x) = f(-x)$ for all x) $\int_{-a}^a f(x)\mathrm{d}x = 2\int_0^a f(x)\mathrm{d}x.$

(viii) If $f(x) \leq g(x)$ for $a \leq x \leq b$, then $\int_a^b f(x)\mathrm{d}x \leq \int_a^b g(x)\mathrm{d}x$

3.2.4 Exercises