Fundamentals Lecture Eight

Basic Trig Identities :

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Basic Trigonometric Identities

(i) Pythagorean Formulae

$$\begin{aligned} \sin^2\theta + \cos^2\theta &= 1 \\ 1 + \cot^2\theta &= \operatorname{cosec}^2\theta \\ \tan^2\theta + 1 &= \sec^2\theta \end{aligned}$$

Question: How are the second and third of these formulae derived from the first?

Example

Show that

$$(\sin\theta + \cos\theta)^2 + (\sin\theta - \cos\theta)^2 = 2$$

Solution

$$\begin{aligned} \text{LHS} &= (\sin\theta + \cos\theta)^2 + (\sin\theta - \cos\theta)^2 \\ &= \\ &= \\ &= \\ &= \\ &= \quad \text{RHS} \quad \square \end{aligned}$$

$$+\sin^2\theta+\cos^2\theta+2\cos\theta\sin\theta$$

$$+\sin^2\theta+\cos^2\theta-2\cos\theta\sin\theta$$

$$2\sin^2\theta+2\cos\theta$$

$$2\left(\sin^2\theta+\cos^2\theta\right)$$

$$2$$

Example

Verify that

$$\frac{1-\sin\theta}{\cos\theta}=\frac{\cos\theta}{1+\sin\theta} \quad \text{Ex1N Q29(c)}$$

Solution Multiply both sides of the equation by $\cos\theta\,(1+\sin\theta)$ to obtain

$$
\begin{aligned}
(1-\sin\theta)(1+\sin\theta) &= \cos\theta\cos\theta \\
1-\sin^2\theta &= \cos^2\theta \\
1 &= \cos^2\theta + \sin^2\theta \\
1 &= 1
\end{aligned}
$$

because $\sin^2\theta + \cos^2\theta = 1$ by the Pythagorean formula

(ii) Sum and Difference of Angle Formulae

$$\begin{aligned}
\sin(A \pm B) &= \sin A \cos B \pm \cos A \sin B \\
\cos(A \pm B) &= \cos A \cos B \mp \sin A \sin B \\
\tan(A \pm B) &= \frac{\tan A \pm \tan B}{1 \mp \tan A \tan B}
\end{aligned}$$

Example Find an exact value for $\cos \frac{7\pi}{12}$

Solution Now, $\frac{7\pi}{12} = \frac{\pi}{3} + \frac{\pi}{4}$

$$\begin{aligned}
\therefore\ \cos \frac{7\pi}{12} &= \cos\left(\frac{\pi}{3} + \frac{\pi}{4}\right) \\
&= \\
&= \\
&= \frac{1-\sqrt{3}}{2\sqrt{2}}
\end{aligned}$$

$$\cos\left(\frac{\pi}{3}\right)\cos\left(\frac{\pi}{4}\right)-\sin\left(\frac{\pi}{3}\right)\sin\left(\frac{\pi}{4}\right)$$

$$\frac{1}{2}\cdot\frac{1}{\sqrt{2}}-\frac{\sqrt{3}}{2}\cdot\frac{1}{\sqrt{2}}$$

Example Find an exact value for $\sin \frac{\pi}{12}$

Solution Now, $\frac{7\pi}{12} = \frac{\pi}{3} - \frac{\pi}{4}$

$$\therefore \sin \frac{\pi}{12} = \sin\left(\frac{\pi}{3} - \frac{\pi}{4}\right)$$

$$=$$

$$=$$

$$=$$

$$= \frac{\sqrt{2}}{4}\left(\sqrt{3} - 1\right)$$

$$\sin\left(\frac{\pi}{3}\right)\cos\left(\frac{\pi}{4}\right)-\cos\left(\frac{\pi}{3}\right)\sin\left(\frac{\pi}{4}\right)$$

$$\frac{\sqrt{3}}{2}\cdot\frac{1}{\sqrt{2}}-\frac{1}{2}\cdot\frac{1}{\sqrt{2}}$$

$$\frac{1}{2\sqrt{2}}\left(\sqrt{3}-1\right)$$

(iii)Double Angle Formulae

$$\begin{aligned}
\sin 2\theta &= 2\sin\theta\cos\theta \\
\cos 2\theta &= \cos^2\theta - \sin^2\theta \\
&= 2\cos^2\theta - 1 \\
&= 1 - 2\sin^2\theta \\
\tan 2\theta &= \frac{2\tan\theta}{1-\tan^2\theta}
\end{aligned}$$

These formula can be obtained by letting $A = B = \theta$ in (ii)

Example Prove that $\frac{1-\cos 2\theta}{\sin 2\theta} = \tan\theta$.

Solution

$$\text{LHS} = \frac{1 - \cos 2\theta}{\sin 2\theta}$$

Now $\cos 2\theta = 1 - 2\sin^2\theta$ (Double Angle Formulae)

$$\therefore \quad \text{LHS} =$$

$$=$$

$$=$$

because $\sin 2\theta = 2\sin\theta\cos\theta$ (Double Angle Formulae)

$$=$$

$$= \quad \text{RHS} \quad \square$$

$$\frac{1-\left(1-2\sin^2\theta\right)}{\sin 2\theta}$$

$$\frac{2\sin^2\theta}{\sin 2\theta}$$

$$\frac{2\sin^2\theta}{2\sin\theta\cos\theta}$$

$$\frac{\sin\theta}{\cos\theta}$$

Half-angle Formulae

$$\begin{aligned}\sin^2\frac{\theta}{2} &= \tfrac{1-\cos\theta}{2}\\ \cos^2\frac{\theta}{2} &= \tfrac{1+\cos\theta}{2}\\ \tan\frac{\theta}{2} &= \tfrac{1-\cos\theta}{\sin\theta}\\ &= \tfrac{\sin\theta}{1+\cos\theta}\end{aligned}$$

Note: From the half-angle formulae

$$\sin\frac{\theta}{2} = \pm\sqrt{\frac{1-\cos\theta}{2}}$$

$$\cos\frac{\theta}{2} = \pm\sqrt{\frac{1+\cos\theta}{2}}$$

The '+' or '-' sign is chosen to be compatible with the known quadrant of $\frac{\theta}{2}$.

Example

1. Explain why $\cos\dfrac{5\pi}{3} = \dfrac{1}{2}$
2. Hence find an exact value for $\cos\frac{5\pi}{6}$ using the half-angle formulae.

Solution We need to use the fact that $\frac{5\pi}{6} = \frac{1}{2}\cdot\left(\frac{5\pi}{3}\right)$ and that

$$\cos^2\left(\frac{\theta}{2}\right) = \frac{1+\cos\theta}{2}$$

$$\Rightarrow \cos\frac{\theta}{2} = \pm\sqrt{\frac{1+\cos\theta}{2}}$$

$$\cos\frac{\theta}{2} = \pm\sqrt{\frac{1+\cos\theta}{2}}$$

Note that $\dfrac{5\pi}{6}$ is in the 2nd quadrant so that $\cos\dfrac{5\pi}{6}$ is negative. Let $\theta = \dfrac{5\pi}{3}$

$$\therefore \quad \cos\frac{5\pi}{6} =$$

$$=$$

$$=$$

$$=$$

$$-\left[\frac{1+\cos\left(\frac{5\pi}{3}\right)}{2}\right]^{1/2}$$

$$-\left[\frac{1}{2}+\frac{1}{4}\right]^{1/2}$$

$$-\left[\frac{3}{4}\right]^{1/2}$$

$$-\frac{\sqrt{3}}{2}$$

Products as Sums or Differences

$$
\begin{aligned}
2\cos A\cos B &= \cos(A+B)+\cos(A-B)\\
2\sin A\sin B &= \cos(A-B)-\cos(A+B)\\
2\sin A\cos B &= \sin(A+B)+\sin(A-B)\\
2\cos A\sin B &= \sin(A+B)-\sin(A-B)
\end{aligned}
$$

These formulae are obtained by the addition or subtraction of the formulae in (ii)

Examples

Write the following products as a sum or difference.

a $2\cos 7x\cos 5x$

b $\cos 4x\sin x$

Solutions

a Let $A = 7x$ and $B = 5x$. Then as

$$2\cos A\cos B = \cos(A+B) + \cos(A-B)$$

we have

$$2\cos 7x \cos 5x = \cos(7x+5x) + \cos(7x-5x])$$
$$= \cos(12x) + \cos(2x)$$

b Let $A = 4x$ and $B = x$. Then as

$$\cos A\sin B = \frac{1}{2}\left[\sin(A+B) - \sin(A-B)\right]$$

we have

$$\cos 4x \sin x = \frac{1}{2}\left[\sin(4x+x) - \sin(4x-x]]\right)$$
$$= \frac{1}{2}\left[\sin(5x) - \sin(3x)\right]$$

Half-angle tangent formula

$$\begin{array}{rrcl} & \text{Let} \quad t & = & \tan\dfrac{x}{2}, \\ \text{then} & \sin x & = & \dfrac{2t}{1+t^2} \\ & \cos x & = & \dfrac{1-t^2}{1+t^2} \\ & \tan x & = & \dfrac{2t}{1-t^2} \end{array}$$

This substitution is useful in integrating rational functions of sine and cosine (MATH 142).

Example Find an expression for $\dfrac{1}{1+\cos x}$ by letting $t = \tan\dfrac{x}{2}$.

Solution Now, $\cos x = \frac{1-t^2}{1+t^2}$. Hence

$$\frac{1}{1+\cos x} =$$

$$=$$

$$=$$

$$\frac{1}{1+\frac{1-t^2}{1+t^2}}$$

$$\frac{1}{\frac{1+t^2}{1+t^2}+\frac{1-t^2}{1+t^2}}$$

$$\frac{1+t^2}{2}.$$

Exercises on Trig Identities

Exercise 1.13.7
pages 44 & 45.

For each of the questions do as many of the subquestions as you require in order to gain mastery of the basic technique.

You *need* to do these exercises!