2.11 PARAMETRIC EQUATIONS AND CURVES

2.11.1 Introduction

Until now we have represented a graph by a single equation involving the two variables x and y. Sometimes it is useful to introduce a third variable to represent a curve in the plane. For instane, when a ball is thrown or a rocket is launched the x coordinate might be given as a function of time after launch, i.e. $x = f(t)$. Similarly the y coordinate might be given by $y = g(t)$.

In these equations the variable t is called a parameter. The equations

$$x = f(t),$$
$$y = g(t),$$

are known as the parametric equations.

For instane, suppose the x and y coordinates of a projectile are given by

$$x = x(t) = 24\sqrt{2}\,t \quad \text{and}$$

$$y = y(t) = -16t^2 + 24\sqrt{2}\,t.$$

From this set of equations we can determine that at time $t = 0$, the object is at point _____. Similarly, at time $t = 1$, the object is at the point __________________ and so on.

In the example, t denotes time, but it might instead denote an angle or the distance a particle has travelled along its path from its starting point.

$(0, 0)$

$\left(24\sqrt{2}, 24\sqrt{2} - 16\right)$

2.11.2 Sketching a Curve

Example

Sketch the trajectory of a particle in the xy-plane described by the parametric equations

$$x = x(t) = t^2 - 4 \quad \text{and}$$

$$y = y(t) = \frac{t}{2} \quad \text{for} \quad -2 \le t \le 3$$

Solution

We select values of t on the given interval and calculate the corresponding x and y coordinates.

t	-2	-1	0	1	2	3
x						
y						

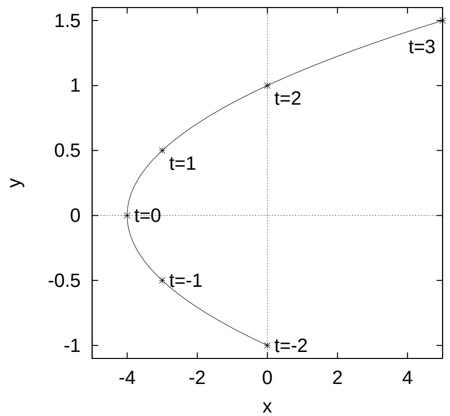

Figure 8: $x = t^2 - 4, y = t/2$

We plot the points (x, y) and connect successive points with a smooth curve. Note that the arrows on the curve indicate the direction of motion. It often happens that different sets of parametric equations describe the same path.

For example, the pair of parametric equations

$$x = 4t^2 - 4 \quad \text{and}$$

$$y = t, \quad -1 \leq t \leq \frac{3}{2}$$

have the same graph as our last example. The difference is that the second pair of equations trace the curve out more *rapidly* (considering t as time) than the first pair.

Examples

1. Sketch the curve given by
   $x = 4t^2 - 4$ and $y = t, \quad -1 \leq t \leq \frac{3}{2}$
   and show that it is identical to the curve found previously.

2. Sometimes a curve given parametrically can be recognised if we convert it to its Cartesian form.

   Find the Cartesian equation of the curve whose parametric equations are
   $x = t^2 - 4$ and $y = \frac{t}{2}, \quad -2 \leq t \leq 3$
   $(x = 4y^2 - 4,\ 0 \leq x \leq 5)$

3. Show that the parametric equations
   $x = 4t^2 - 4$ and $y = t, \quad -1 \leq t \leq \frac{3}{2}$
   can also be described by the Cartesian equation
   $x = 4y^2 - 4, \quad 0 \leq x \leq 5$

4. Find the Cartesian equation of the curve whose parametric equations are given by

$$x = 2\cos t \text{ and}$$
$$y = 2\sin t, \quad 0 \leq t \leq 2\pi.$$

$(x^2 + y^2 = 4)$

1. Sketch the curve given by
$x = 4t^2 - 4$ and $y = t$, $-1 \leq t \leq \frac{3}{2}$
and show that it is identical to the curve found previously.

The curve is $x = 4y^2 - 4$, where the end points are $(0, -1)$ (when $t = -1$) and $\left(5, \dfrac{3}{2}\right)$ (when $t = \dfrac{3}{2}$). This is identical to the curve found previously.

(I have not sketched the curve. If this were an exam question you would not score full marks if your answer did not include the sketch).

2. Sometimes a curve given parametrically can be recognised if we convert it to its Cartesian form.

Find the Cartesian equation of the curve whose parametric equations are $x = t^2 - 4$ and $y = \frac{t}{2}$, $-2 \le t \le 3$

$$x = t^2 - 4, \quad y = \frac{t}{2}$$

$$x = (2y)^2 - 4$$

$$x = 4y^2 - 4$$

$y = \dfrac{t}{2}$ is an *increasing* function of t. The end points are therefore $y = -1$ (when $t = -2$) and $y = \dfrac{3}{2}$ when $t = 3$. The curve therefore goes from the point $(0, -1)$ to the point $\left(5, \dfrac{3}{2}\right)$.

3. Show that the parametric equations
$x = 4t^2 - 4$ and $y = t$, $-1 \leq t \leq \frac{3}{2}$
can also be described by the Cartesian
equation $x = 4y^2 - 4$, $0 \leq x \leq 5$

$$x = 4t^2 - 4 \quad y = t$$

$$x = 4\,(y)^2 - 4$$

$$x = 4y^2 - 4.$$

$y = t$ is an *increasing* function of t. The end points are therefore $y = -1$ (when $t = -1$) and $y = \frac{3}{2}$ when $t = \frac{3}{2}$. The curve therefore goes from the point $(0, -1)$ to the point $\left(5, \frac{3}{2}\right)$.

4. Find the Cartesian equation of the curve whose parametric equations are given by
$x = 2\cos t$ and $y = 2\sin t$, $0 \le t \le 2\pi$.

$$x = 2\cos t \qquad\qquad y = 2\sin t$$
$$x^2 = 4\cos^2 t \qquad\qquad y^2 = 4\sin t$$
$$x^2 + y^2 = 4\cos^2 t + 4\sin^2 t$$
$$x^2 + y^2 = 4\left(\cos^2 t + \sin^2 t\right)$$
$$x^2 + y^2 = 4.$$

Note that as g increase over the range $0 \le t \le 2\pi$ we move from the initial point $(2, 0)$ all around the circle back to the initial $(2, 0)$ when $t = 2\pi$.

2.11.3 Derivatives of Parametric Curves

The parametric equations

$$x = x(t) \quad \text{and} \quad y = y(t)$$

can be differentiated with respect to t to give $\dfrac{dx}{dt}$ and $\dfrac{dy}{dt}$ respectively.

We can use these derivatives and the chain rule to find $\dfrac{dy}{dx}$.

$$\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}, \quad \frac{dx}{dt} \neq 0$$

Examples

1. Given the curve
   $x = t^2 - 4$ and $y = \dfrac{t}{2}$, $\quad -2 \leq t \leq 3$
   then $\dfrac{dx}{dt} =$ __
   and $\dfrac{dy}{dt} =$ __ .
   Hence
   $$\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \_\_$$

$2t$

$\frac{1}{2}$

$\frac{1}{4t}$

2. Given the curve
$x = 2 + \sec t$ and $y = 1 + 2\tan t$ find the slope of the curve at the point $t = \dfrac{\pi}{6}$.

$$\begin{aligned} x &= 2 + \sec t \\ &= 2 + \frac{1}{\cos t} \\ \frac{\mathrm{d}x}{\mathrm{d}t} &= \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{\cos t}\right) \\ &= \frac{\cos t \frac{\mathrm{d}}{\mathrm{d}t}(1) - 1\frac{\mathrm{d}}{\mathrm{d}t}(\cos t)}{(\cos t)^2} \\ &= \frac{\sin t}{\cos^2 t} \\ &= \tan t \sec t \end{aligned}$$

$$\begin{aligned} y &= 1 + 2\tan t \\ \frac{\mathrm{d}y}{\mathrm{d}t} &= 2\frac{\mathrm{d}}{\mathrm{d}t}\tan t \\ &= 2\sec^2 t \end{aligned}$$

$$
\begin{aligned}
\frac{dy}{dx} &= \frac{dy}{dt} \cdot \frac{dt}{dx} \\
&= \frac{dy}{dt} \cdot \frac{1}{\frac{dx}{dt}} \\
&= 2\sec^2 t \cdot \frac{1}{\sec t \tan t} \\
&= \frac{2\sec t}{\tan t} \\
&= \frac{2}{\cos t \tan t} \\
&= \frac{2}{\sin t}
\end{aligned}
$$

When $t = \frac{\pi}{6}$ we have

$$
\begin{aligned}
\frac{dy}{dx} &= \frac{2}{\sin \frac{\pi}{6}} \\
&= 4.
\end{aligned}
$$

2.11.4 Finding $\frac{d^2y}{dx^2}$

Consider again our parametric equations $x = t^2 - 4$ and $y = \frac{t}{2}$, $-2 \leq t \leq 3$.

We have found that $\frac{dy}{dx} = \frac{1}{4t}$.

Now,

$$
\begin{aligned}
\frac{d^2y}{dx^2} &= \underline{\frac{d}{dx}\left(\frac{dy}{dx}\right)} \\
&= \underline{\frac{d}{dx}\left(\frac{1}{4t}\right)}
\end{aligned}
$$

but we cannot differentiate a function of t directly with respect to x.

We have to apply the chain rule again.

$$\frac{d^2y}{dx^2} = \underline{\frac{d}{dx}\left(\frac{dy}{dx}\right)}$$
$$= \underline{\frac{d}{dt}\left(\frac{dy}{dx}\right)\cdot\frac{dt}{dx}}$$

So,

$$\frac{d^2y}{dx^2} = \underline{\frac{d}{dx}\left(\frac{dy}{dx}\right)}$$
$$= \underline{\frac{d}{dx}\left(\frac{1}{4t}\right)}$$
$$= \underline{\frac{d}{dt}\left(\frac{1}{4t}\right)\cdot\frac{dt}{dx}}$$
$$= \underline{\frac{d}{dt}\left(\frac{1}{4t}\right)\cdot\frac{1}{\frac{dx}{dt}}}$$
$$= \underline{\qquad\qquad}$$
$$= \underline{\qquad}.$$

$$-\frac{1}{4t^2}\cdot\frac{1}{2t}$$
$$-\frac{1}{8t^3}$$

Example

Find, $\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2}$ given $x = t - \sin t$ and $y = 1 - \cos t$.

$$x = t - \sin t \qquad\qquad y = 1 - \cos t$$

$$\begin{aligned}\frac{\mathrm{d}x}{\mathrm{d}t} &= \frac{\mathrm{d}}{\mathrm{d}t}(t - \sin t) & \frac{\mathrm{d}y}{\mathrm{d}t} &= \frac{\mathrm{d}}{\mathrm{d}t}(1 - \cos t)\\ &= 1 - \cos t & &= \sin t\end{aligned}$$

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{\mathrm{d}y}{\mathrm{d}t} \cdot \frac{\mathrm{d}t}{\mathrm{d}x} \\ &= \frac{\mathrm{d}y}{\mathrm{d}t} \cdot \frac{1}{\frac{\mathrm{d}x}{\mathrm{d}t}} \qquad = \frac{\sin t}{1 - \cos t}\end{aligned}$$

$$\begin{aligned}\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} &= \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)\\ &= \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\sin t}{1-\cos t}\right)\\ &= \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\sin t}{1-\cos t}\right)\cdot\frac{\mathrm{d}t}{\mathrm{d}x}\end{aligned}$$

$$= \frac{(1-\cos t)\,\frac{\mathrm{d}}{\mathrm{d}t}(\sin t) - \sin t\,\frac{\mathrm{d}}{\mathrm{d}t}(1-\cos t)}{(1-\cos t)^2}$$

$$\times \frac{1}{1-\cos t}$$

$$= \frac{(1-\cos t)\cos t - \sin t\,(\sin t)}{(1-\cos t)^3}$$

$$= \frac{\cos t - \cos^2 t - \sin^2 t}{(1-\cos t)^3}$$

$$= \frac{\cos t - 1}{(1-\cos t)^3} = -\frac{1}{(1-\cos t)^2}$$

2.11.5 Revision Questions

The following questions are about the key ideas in this section.

1. Suppose that $y = f(t)$ and $x = g(t)$. What is $\dfrac{\mathrm{d}y}{\mathrm{d}x}$?
2. **Exercise** 2.11.6