## 2.2 LIMITS

### 2.2.1 The basics

*Definition:* If $\lim_{x \to a} f(x) = L$, then as $\underline{x}$ approaches $\underline{a}$, f($x$) approaches $\underline{L}$, where $\underline{L}$ is a finite number.



**Example**

Consider $f(x) = x + 2$. As $x$ approaches 2, what happens to $f(x)$?

| $x$ | 1.95 | 1.99 | 1.995 | 2 | 2.005 | 2.01 | 2.05 |
|---|---|---|---|---|---|---|---|
| $f(x)$ | | | | | | | |

The table shows that, as $x$ approaches 2 from either the left or right of 2, $\underline{f(x) \text{ approaches } 4}$; i.e. we can make $\underline{f(x)}$ as close as we like to $\underline{4}$ by making $x$ sufficiently close to $\underline{2}$.

We write

$$\lim_{x\to 2} f(x) = \underline{\lim_{x\to 2}(x+2)}$$
$$= \underline{4}$$

When looking at this limit we are not concerned what happens at $x = 2$ only what is happening to $f(x)$ as $x$ approaches 2.

The limit of a function as $x$ approaches a point does not depend on the value of the function at the point.

For example, consider the function

$$y = \begin{cases} x, & x \neq 2 \\ 4, & x = 2 \end{cases}$$

$\lim_{x\to 2} f(x) = 2$ but $f(2) = 4$.

However, if $f(x)$ is a 'nice' function we can find $\lim_{x\to a} f(x)$ by substituting $x = a$ into $f(x)$.

**Examples**

1. $\lim_{x\to 2}(x+2) =$_
2. $\lim_{x\to -3} x^3 =$___

4

-27

### 2.2.2 Theorems on Limits

If $\lim_{x \to a} f(x) = L$ and $\lim_{x \to a} g(x) = M$, then

1. If $f(x) = k$, where $k$ is a constant, then $\lim_{x \to a} f(x) = \underline{k}$.

   For example, figure 4 shows the function $f(x) = 3$. Observe that regardless of the value of $a$, as $x \to a$, $\underline{f(x) = 3}$, therefore $\underline{\lim_{x \to a} f(x) = 3}$.

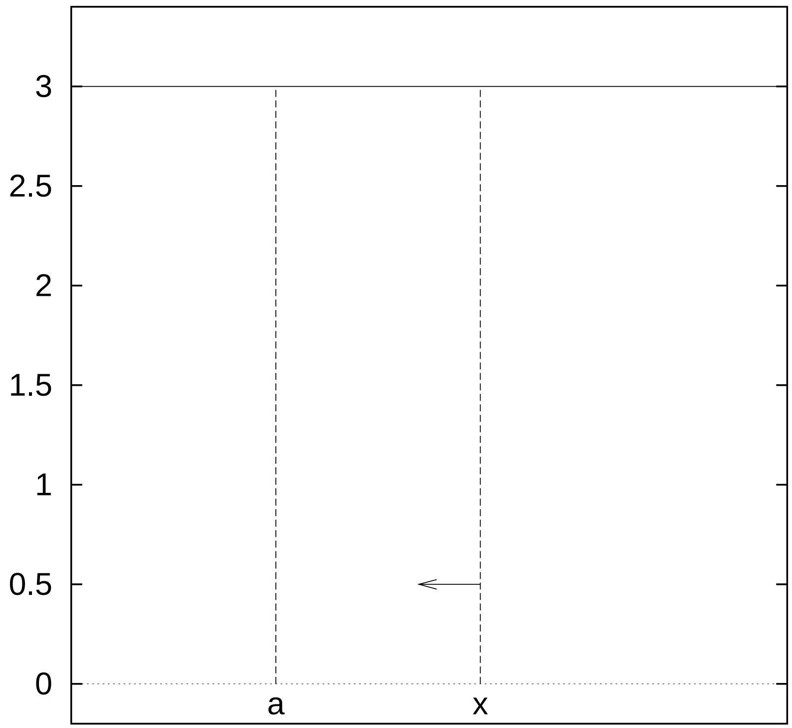


Figure 4: The function $f(x) = 3$.

2.

$$\lim_{x\to a} [f(x) \pm g(x)] = \underline{\lim_{x\to a} f(x) \pm \lim_{x\to a} g(x)}$$
$$= \underline{L \pm M}.$$

For example

$$\lim_{x\to 2} (x^2 - 3x + 5) = \lim_{x\to 2} x^2 - \lim_{x\to 2} 3x + \lim_{x\to 2} 5$$
$$=$$
$$= \underline{3}$$

$(2)^2 - (3 \times 2) + 5$

3.

$$\lim_{x \to a} [f(x).g(x)] = \underline{\lim_{x \to a} f(x) \times \lim_{x \to a} g(x)}$$
$$= \underline{LM}.$$

For example,

$$\lim_{x \to -1} 2x(x^2 - 4) = \lim_{x \to -1} 2x \times \lim_{x \to -1} (x^2 - 4)$$
$$=$$
$$=$$
$$= \underline{6}.$$

$$2\,(-1) \times \left((-1)^2 - 4\right)$$

$$-2 \times -3$$

4.

$$\lim_{x \to a} \left[\frac{f(x)}{g(x)}\right] = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$$

$$= \frac{L}{M} \quad \text{for } M \neq 0.$$

For example

$$\lim_{x \to 3} \left[\frac{x^2+2}{x+1}\right] = \frac{\lim_{x\to 3}(x^2+2)}{\lim_{x\to 3}(x+1)}$$

$$=$$

$$= \_\_\,.$$

$$\frac{3^2+2}{3+1}$$
$$\frac{11}{4}$$

### 2.2.3 Limits of Rational Functions

*Definition* A *rational function* is a function that can be expressed in the form $f(x) = \dfrac{p(x)}{q(x)}$ where $p(x)$ and $q(x)$ are polynomials.

For example, $\dfrac{x^5 - 2x^2 + 1}{x^2 - 4}$, $\dfrac{x}{x+1}$, $\dfrac{1}{x^5}$ etc.

Consider the following limit (using Theorem 4)

$$\lim_{x\to 2}\frac{5x^3+4}{x-3} = \frac{\lim_{x\to 2} 5x^3+4}{\lim_{x\to 2} x-3}$$

$$= \underline{\qquad\qquad}$$

$$= \underline{-44}$$

Now consider $\lim_{x\to 1}\frac{1-x^2}{1-x}$. Theorem 4 is not applicable because $\underline{\lim_{x\to 1}(1-x)=0}$.

However, if both the numerator and denominator of a rational function approach zero as $x$ approaches $a$, then the numerator and denominator will have a common factor $(x-a)$ and the limit can often be calculated by first cancelling the common factors.

For example,

$$\lim_{x\to 1}\frac{1-x^2}{1-x} = \frac{\lim_{x\to 1}(1-x^2)}{\lim_{x\to 1}(1-x)}$$

$$= \underline{\quad}$$

Now

$$\begin{aligned}\lim_{x\to 1}\frac{1-x^2}{1-x} &= \lim_{x\to 1}\frac{(1-x)(1+x)}{\cancel{(1-x)}}\\ &= \lim_{x\to 1}(1+x)\\ &= \lim_{x\to 1}(1+x)\\ &= 2.\end{aligned}$$

**Examples** Find

1. $\displaystyle\lim_{x\to 3}\frac{x^2-6x+9}{x-3}$.

2. $\displaystyle\lim_{x\to -4}\frac{2x+8}{x^2+x-12}$.

$$\lim_{x \to 3} \frac{x^2 - 6x + 9}{x - 3} = \lim_{x \to 3} \frac{(x-3)^2}{(x-3)} = \lim_{x \to 3} (x - 3) = 0.$$

$$\lim_{x \to -4} \frac{2x + 8}{x^2 + x - 12} = \lim_{x \to -4} \frac{2(x+4)}{(x+4)(x-3)} = \lim_{x \to -4} \frac{2}{x - 3} = \frac{-2}{7}$$

If the denominator of a rational function has a limit of zero and the numerator has a limit that is not zero, then the limit of the rational function approaches infinity or we say that the limit does not exist.

For example $\lim_{x\to 4} \frac{1}{x-4} = \frac{1}{0} \to \infty$ or $\lim_{x\to 4} \frac{1}{x-4}$ D.N.E..

### 2.2.4 Limit of Rational Functions as $x \to \pm\infty$

Consider the graphs of the polynomials $x$, $x^2$, $x^3$ and $x^4$. What happens when $x \to +\infty$ and when $x \to -\infty$?

We have

$$\lim_{x\to+\infty} x^n = +\infty \quad n = 1, 2, 3 \ldots$$

$$\lim_{x\to-\infty} x^n = \begin{cases} +\infty, & n = 2, 4, 6 \ldots \\ -\infty, & n = 1, 3, 5 \ldots \end{cases}$$

Consider the limit $\lim_{x\to+\infty} \frac{3x+5}{6x-8}$.

Now, $\lim_{x \to +\infty} (3x + 5) = \underline{\infty}$ and $\lim_{x \to +\infty} (6x - 8) = \underline{\infty}$ i.e. the numerator and denominator <u>both</u> approach infinity as $x$ approaches infinity.

We can solve this type of limit by dividing <u>each term</u> in both the numerator and denominator by the <u>highest power of $x$</u> that occurs in the function.

In our example $\lim_{x \to +\infty} \frac{3x+5}{6x-8}$ the highest power of $x$ that occurs is $x^1 = x$.

Therefore

$$\lim_{x \to +\infty} \frac{3x+5}{6x-8} =$$

$$=$$

$$=$$

**Example**

$$\lim_{x \to \infty} \frac{4x^2 - x}{2x^3 - 5} =$$

$$=$$

$$=$$

$$=$$

$$\frac{\frac{1}{x}}{\frac{1}{x}} \cdot \frac{3x+5}{6x-8}$$

$$\frac{\frac{3x}{x}+\frac{5}{x}}{\frac{6x}{x}-\frac{8}{x}}$$

$$\frac{3+\frac{5}{x}}{6-\frac{8}{x}} = \frac{1}{2}$$

$$\frac{\frac{1}{x^3}}{\frac{1}{x^3}} \cdot \frac{4x^2-x}{2x^3-5}$$

$$\frac{\frac{4x^2}{x^3}-\frac{x}{x^3}}{\frac{2x^3}{x^3}-\frac{5}{x^3}}$$

$$\frac{\frac{4}{x}-\frac{1}{x^2}}{2-\frac{5}{x^3}}$$

$$\frac{0-0}{2-0} = 0.$$

### 2.2.5 Exercises

pages 11 & 12.

### 2.2.6 Revision of key ideas

The following questions are about the key ideas in this section.

1. What do we mean by the expression $\lim_{x \to a} f(x) = L$?
2. What are the following limits?
   (a) $\lim_{x \to 0} x^n$ $(n > 0)$
   (b) $\lim_{x \to +\infty} x^n$ $(n > 0)$
   (c) $\lim_{x \to -\infty} x^n$ $(n > 0)$
   (d) $\lim_{x \to 0} \dfrac{1}{x}$