2.6 INVERSE TRIGONOMETRIC FUNCTIONS

Before the next lecture you should read sections 2.7.1–2.7.3 — I will *assume* that you have done so.

2.6.1 The Inverse Sine Function

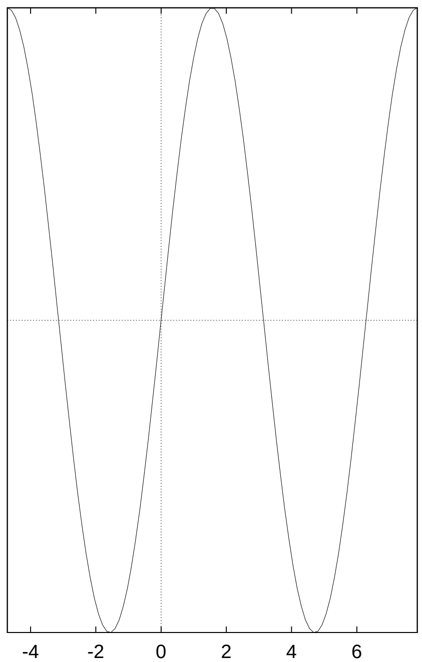
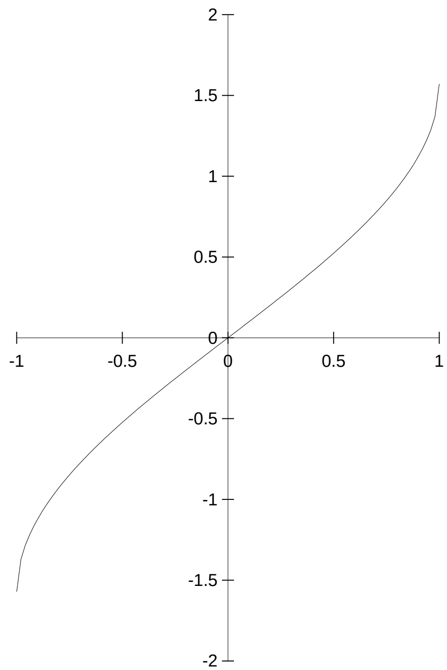


Figure 9: $y = \sin x$

The graph of $y = \sin x$ shows that $\sin x$ is a many-to-one function i.e. there are many x-values that will give the same y-value. To find the inverse of $\sin x$ we need to *restrict its domain* so that we have a 1-1 function.

We will restrict the domain of $y = \sin x$ to $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$.

The **inverse sine function** is denoted by $y = \sin^{-1} x$ or $y = \arcsin x$.



The Graph of $y = \sin^{-1} x$

From the graph note:

1. $\operatorname{Dom} \sin^{-1} x = \underline{[-1, 1]}$
2. $\operatorname{Range} \sin^{-1} x = \underline{\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]}$

Figure 10: $y = \sin^{-1} x$

Note: $\operatorname{Dom} \sin x = \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ and $\operatorname{Range} \sin x = [-1, 1]$.

Let $y = \sin^{-1} x$ and take sine of both sides

$\therefore \underline{\sin y = \sin\left(\sin^{-1} x\right) = x}$

Definition

If $y = \sin^{-1} x$, then $\underline{x = \sin y}$, for $\underline{-1 \le x \le 1}$ and $\underline{-\frac{\pi}{2} \le y \le \frac{\pi}{2}}$.

In words we can describe $\sin^{-1} x$ as <u>the angle whose sin is x</u>.

Note $\sin^{-1} x$ is the <u>inverse</u> of $\sin x$ *not* the reciprocal.

$$\frac{1}{\sin x} = [\sin(x)]^{-1}$$

Examples

Find exactly:

(a) $\sin^{-1}\left(\frac{1}{2}\right)$ (b) $\sin^{-1}\left(-\frac{1}{2}\right)$

(c) $\sin^{-1}\left(\frac{\sqrt{3}}{2}\right)$ (d) $\sin^{-1} 1$

Find exactly: (a) $\sin^{-1}\left(\frac{1}{2}\right)$

$$\text{Let} \quad y = \sin^{-1}\left(\frac{1}{2}\right), \qquad -\frac{\pi}{2} \le y \le \frac{\pi}{2}$$

$$\therefore \quad \sin y = \frac{1}{2}$$

$$\therefore \quad y = \frac{\pi}{6}$$

(b) Find exactly $\sin^{-1}\left(-\frac{1}{2}\right)$

$$\text{Let} \quad y = \sin^{-1}\left(-\frac{1}{2}\right), \qquad -\frac{\pi}{2} \le y \le$$

$$\therefore \quad \sin y = -\frac{1}{2}$$

Therefore y is in the fourth quadrant

$$\therefore \quad y = -\frac{\pi}{6}$$

Find exactly (c) $\sin^{-1}\left(\frac{\sqrt{3}}{2}\right)$

$$\text{Let} \quad y = \sin^{-1}\left(\frac{\sqrt{3}}{2}\right), \qquad -\frac{\pi}{2} \leq y \leq$$

$$\therefore \quad \sin y = \frac{\sqrt{3}}{2}$$

$$\therefore \quad y = \frac{\pi}{3}$$

Find exactly (d) $\sin^{-1} 1$

$$\text{Let} \quad y = \sin^{-1}(1), \qquad -\frac{\pi}{2} \le y \le \frac{\pi}{2}$$

$$\therefore \quad \sin y = 1$$

$$\therefore \quad y = \frac{\pi}{2}$$

2.6.2 The Inverse Cosine Function

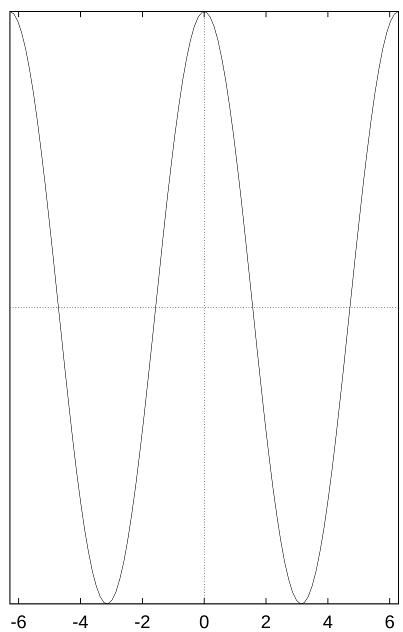


Figure 11: $y = \cos x$

Like the sine function, the cosine function is not 1-1. We can make a 1-1 function by restricting the domain to $\underline{0 \le x \le \pi}$.

The **inverse cosine function** is denoted by $\underline{y = \cos^{-1} x}$ or $\underline{y = \arccos x}$.

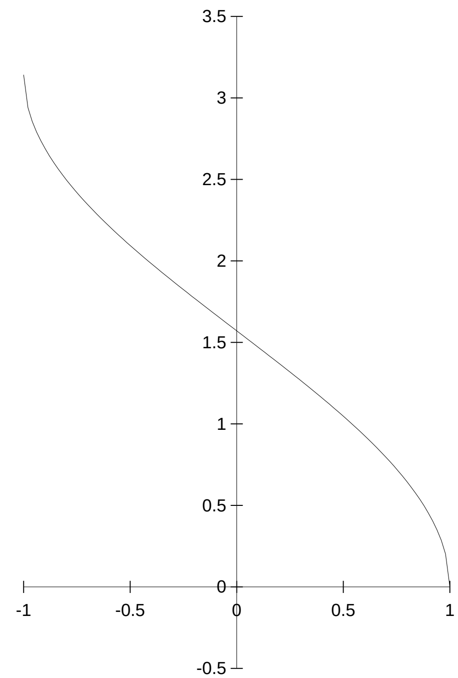


Figure 12: $y = \cos^{-1} x$

The Graph of $y = \cos^{-1} x$

From the graph note

1. $\text{Dom} \cos^{-1} x = \underline{[-1, 1]}$

2. $\text{Range} \cos^{-1} x = \underline{[0, \pi]}$

Note: $\text{Dom} \cos x = [0, \pi]$ and $\text{Range} \cos x = [-1, 1]$.

Let $y = \cos^{-1} x$ and take cosine of both sides

$\therefore \underline{\cos y = \cos\left(\cos^{-1} x\right) = x}$

Definition

If $y = \cos^{-1} x$, then $\underline{x = \cos y}$, for $\underline{-1 \leq x \leq 1}$ and $\underline{0 \leq y \leq \pi}$.

In words we can describe $\cos^{-1} x$ as the angle whose cosine is x.

Examples

Find exactly:

(a) $\cos^{-1}\left(\frac{1}{2}\right)$ (b) $\cos^{-1}\left(\frac{-1}{\sqrt{2}}\right)$

(c) $\cos^{-1}\left(\frac{\sqrt{3}}{2}\right)$ (d) $\cos^{-1} 0$

Find exactly: (a) $\cos^{-1}\left(\frac{1}{2}\right)$

Let $\quad y = \cos^{-1}\left(\dfrac{1}{2}\right)$

$$\therefore \quad \cos y = \frac{1}{2} \quad 0 \leq y \leq \pi$$

$$\therefore \quad y = \frac{\pi}{3}$$

Find exactly: (b) $\cos^{-1}\left(\frac{-1}{\sqrt{2}}\right)$

$$\text{Let} \quad y = \cos^{-1}\left(\frac{-1}{\sqrt{2}}\right)$$

$$\therefore \quad \cos y = -\frac{1}{\sqrt{2}} \quad 0 \leq y \leq \pi$$

cosine is negative so y is in the second quadrant.

$$\therefore \quad y = \frac{3\pi}{4}$$

Find exactly: (c) $\cos^{-1}\left(\frac{\sqrt{3}}{2}\right)$

$$\text{Let} \quad y = \cos^{-1}\left(\frac{\sqrt{3}}{2}\right)$$

$$\therefore \quad \cos y = \frac{\sqrt{3}}{2} \quad 0 \leq y \leq \pi$$

$$\therefore \quad y = \frac{\pi}{6}$$

Find exactly: (d) $\cos^{-1} 0$

$$\text{Let} \quad y = \cos^{-1}(0)$$
$$\therefore \quad \cos y = 0 \quad 0 \le y \le \pi$$
$$\therefore \quad y = \frac{\pi}{2}$$

2.6.3 The Inverse Tangent Function

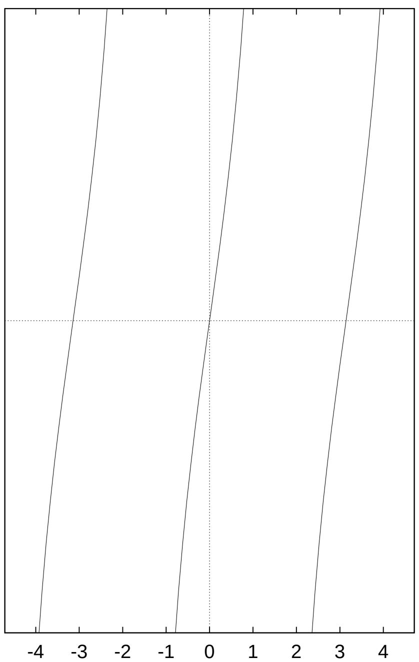


Figure 13: $y = \tan x$

We restrict the tangent function to the domain $\underline{-\frac{\pi}{2} < x < \frac{\pi}{2}}$ so that we have a 1-1 function. The **inverse tangent function** is denoted by $\underline{y = \tan^{-1} x}$ or $\underline{y = \arctan x}$.

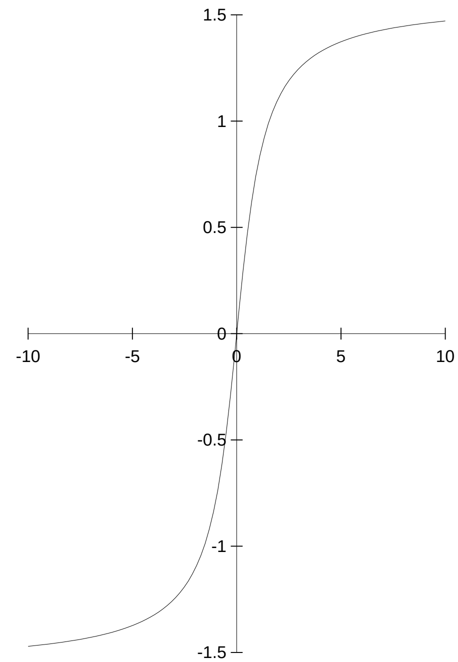


Figure 14: $y = \tan^{-1} x$

The Graph of $y = \tan^{-1} x$

From the graph note

1. $\operatorname{Dom} \tan^{-1} x = \underline{\mathbb{R}}$

2. $\operatorname{Range} \tan^{-1} x = \underline{\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)}$

Note: $\operatorname{Dom} \tan x = \left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$ and $\operatorname{Range} \tan x = \mathbb{R}$.

Let $y = \tan^{-1} x$ and take tangent of both sides

$\therefore \underline{\tan y = \tan(\tan^{-1} x) = x}$.

Definition

If $y = \tan^{-1} x$, then $\underline{x = \tan y}$, for $\underline{-\infty < x < \infty}$ and $\underline{-\frac{\pi}{2} < y < \frac{\pi}{2}}$.

In words we can describe $\tan^{-1} x$ as the angle whose tangent is x.

Examples

Find exactly:

(a) $\tan^{-1} 1$

(b) $\tan^{-1}(-\sqrt{3})$

(c) $\tan^{-1}(-1)$

(d) $\tan^{-1}\left(\frac{1}{\sqrt{3}}\right)$

Find exactly: (a) $\tan^{-1} 1$

$$\text{Let} \quad y = \tan^{-1} 1$$

$$\therefore \quad \tan y = 1 \qquad -\frac{\pi}{2} < y < \frac{\pi}{2}$$

$$\therefore \quad y = \frac{\pi}{4}$$

Find exactly: (b) $\tan^{-1}(-\sqrt{3})$

$$\text{Let} \quad y = \tan^{-1} -\sqrt{3}$$

$$\therefore \quad \tan y = -\sqrt{3} \quad -\frac{\pi}{2} < y < \frac{\pi}{2}$$

tangent is negative in the fourth quadrant

$$\therefore \quad y = -\frac{\pi}{3}$$

Find exactly: (c) $\tan^{-1}(-1)$

$$\text{Let} \quad y = \tan^{-1} -1$$

$$\therefore \quad \tan y = -1 \qquad -\frac{\pi}{2} < y < \frac{\pi}{2}$$

tangent is negative in the fourth quadrant

$$\therefore \quad y = -\frac{\pi}{4}$$

Find exactly: (d) $\tan^{-1}\left(\frac{1}{\sqrt{3}}\right)$

$$\text{Let} \quad y = \tan^{-1}\frac{1}{\sqrt{3}}$$

$$\therefore \quad \tan y = \frac{1}{\sqrt{3}} \qquad -\frac{\pi}{2} < y < \frac{\pi}{2}$$

$$\therefore \quad y = \frac{\pi}{6}$$

Further properties of the inverse trigonometric functions

$$\sin^{-1}(\sin x) = x \quad \text{if } -\frac{\pi}{2} \le x \le \frac{\pi}{2}$$
$$\sin(\sin^{-1} x) = x \quad \text{if } -1 \le x \le 1$$
$$\cos^{-1}(\cos x) = x \quad \text{if } \; 0 \le x \le \pi$$
$$\cos(\cos^{-1} x) = x \quad \text{if } -1 \le x \le 1$$
$$\tan^{-1}(\tan x) = x \quad \text{if } -\frac{\pi}{2} \le x \le \frac{\pi}{2}$$
$$\tan(\tan^{-1} x) = x \quad \text{if } -\infty < x < \infty$$

Example Simplify the function $\cos\left(\sin^{-1} x\right)$ for $|x| \le 1$.

Solution

Method 1

Recall $\sin^2\theta + \cos^2\theta = 1$

$$\Rightarrow \cos\theta = \pm\sqrt{1-\sin^2\theta}$$

Let $\theta = \sin^{-1} x$

$$\therefore \quad \cos\left(\sin^{-1} x\right) = \pm\sqrt{1-\sin^2\left(\sin^{-1} x\right)}$$

Now $-\dfrac{\pi}{2} \leq \sin^{-1} x \leq \dfrac{\pi}{2}$

and $\cos > 0$ in this interval

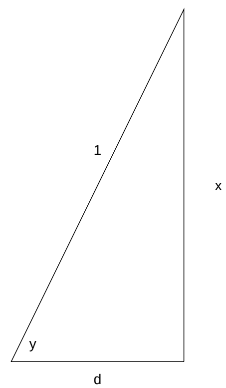
also $\sin\left(\sin^{-1} x\right) = x, \quad \text{if} \quad |x| \leq 1$

$$\therefore \quad \cos\left(\sin^{-1} x\right) = \sqrt{1-x^2}$$

Example Simplify the function $\cos\left(\sin^{-1} x\right)$ for $|x| \leq 1$.

Solution Method 2.

$$y = \sin^{-1} x \Longrightarrow \sin(y) = x.$$



From the triangle we have

$$d = \sqrt{1 - x^2}$$

$$\cos y = \frac{\sqrt{1 - x^2}}{1}$$

$$\cos\left(\sin^{-1} x\right) = \sqrt{1 - x^2}.$$

Take the positive root for the same reasons as in method 1.

2.6.4 Revision Questions

The following questions are about the key ideas in this section.

1. Sketch the following graphs: (a) $y = \sin^{-1} x$ (b) $y = \cos^{-1} x$ (c) $y = \tan^{-1} x$.
2. What are the domain and range of: (a) $y = \sin^{-1} x$ (b) $y = \cos^{-1} x$ (c) $y = \tan^{-1} x$.
3. Why do we need to restrict the domain of the inverse trigonometric functions?

2.6.5 Exercises

1. Simplify the following functions.
2. Evaluate (if possible) the following expressions (don't use a calculator!)