

2.7 INVERSE HYPERBOLIC FUNCTIONS

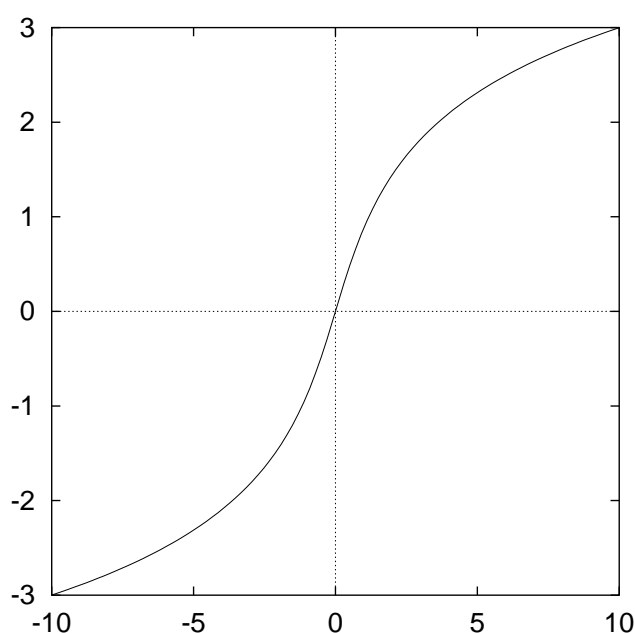
You were asked to read through sections 2.7.1–2.7.3 before the lecture. I will not go through this material in the lecture, but will assume that you know it. Equations (1)–(3) are particularly useful and you are advised to make a note of them.

$$\sinh^{-1} x = \ln \left[x + \sqrt{x^2 + 1} \right], x \in \mathbb{R}$$

$$\cosh^{-1} x = \ln \left[x + \sqrt{x^2 - 1} \right], x \geq 1$$

$$\tanh^{-1} x = \frac{1}{2} \ln \left| \frac{1+x}{1-x} \right|, -1 < x < 1$$

2.7.1 The Inverse Hyperbolic Sine Function



The Graph of
 $y = \sinh^{-1} x$ (or
 $y = \operatorname{arcsinh} x$)

Figure 15: $y = \sinh^{-1} x$

From the graph note

1. $\operatorname{Dom} \sinh^{-1} x = \underline{\mathbb{R}}$
2. $\operatorname{Range} \sinh^{-1} x = \underline{\mathbb{R}}$

Definition If $y = \sinh^{-1} x$, then $x = \sinh y$, for $x \in \underline{\mathbb{R}}$ and $y \in \underline{\mathbb{R}}$.

Properties

$$\sinh (\sinh^{-1} x) = x, x \in \mathbb{R}$$

$$\sinh^{-1} (\sinh x) = x, x \in \mathbb{R}$$

$$\sinh^{-1} x = \ln \left[x + \sqrt{x^2 + 1} \right] \quad (1)$$

2.7.2 The Inverse Hyperbolic Cosine Function

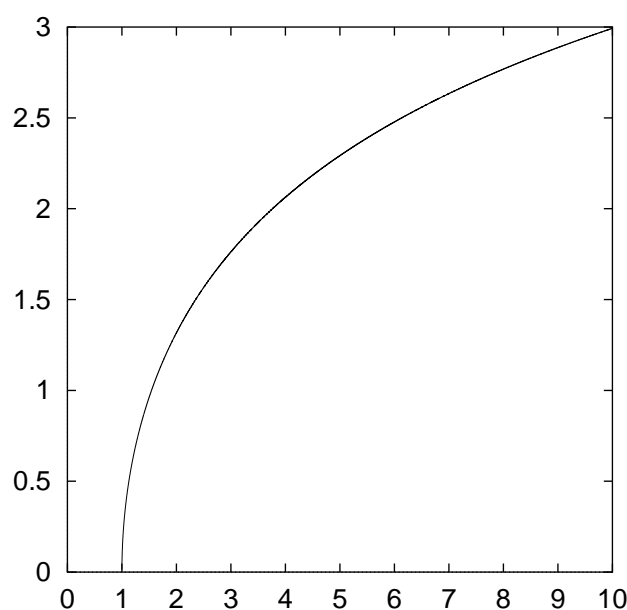


Figure 16: The graph $y = \cosh^{-1} x$

From the graph note

1. $\text{Dom } \cosh^{-1} x = \underline{[1, \infty)}$
2. $\text{Range } \cosh^{-1} x = \underline{[0, \infty)}$

Definition

If $y = \cosh^{-1} x$, then $x = \cosh y$, for $x \geq 1$ and $y \geq 0$.

Properties

$$\cosh (\cosh^{-1} x) = x, x \geq 1$$

$$\cosh^{-1} (\cosh x) = x, x \geq 0$$

$$\cosh^{-1} x = \ln \left[x + \sqrt{x^2 - 1} \right] \quad (2)$$

2.7.3 The Inverse Hyperbolic Tangent Function

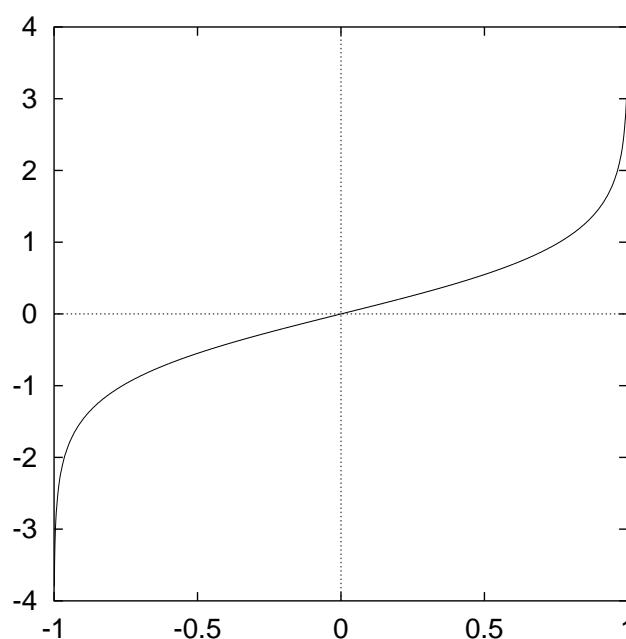


Figure 17: The graph $y = \tanh^{-1} x$

From the graph note

1. $\text{Dom } \tanh^{-1} x = \underline{(-1, 1)}$
2. $\text{Range } \tanh^{-1} x = \underline{\mathbb{R}}$

Definition If $y = \tanh^{-1} x$, then
 $x = \tanh y$, for $-1 < x < 1$ and $y \in \mathbb{R}$.

Properties

$$\tanh(\tanh^{-1} x) = x, -1 < x < 1$$

$$\tanh^{-1}(\tanh x) = x, x \in \mathbb{R}$$

$$\tanh^{-1} x = \frac{1}{2} \ln \left| \frac{1+x}{1-x} \right|, -1 < x < 1$$

(3)

2.7.4 Examples

1. Evaluate:

(a) $\sinh 1.275$ (b) $\tanh^{-1} 0.75$

2. Prove that

$$\sinh^{-1} x = \ln [x + \sqrt{x^2 + 1}]$$

(Ex 2.7.6 Q2)

(1a) Evaluate $\sinh 1.275$.

(1b) Evaluate $\tanh^{-1} 0.75$

$$\sinh x = \frac{1}{2} (e^x - e^{-x})$$

$$\begin{aligned}\sinh 1.275 &= \frac{1}{2} (e^{1.275} - e^{-1.275}) \\ &\approx 1.649\end{aligned}$$

$$\tanh^{-1} x = \frac{1}{2} \ln \left| \frac{1+x}{1-x} \right|$$

$$\begin{aligned} \tanh^{-1} 0.75 &= \frac{1}{2} \ln \left| \frac{1+0.75}{1-0.75} \right| \\ &\approx 0.973 \end{aligned}$$

2. Prove that

$$\sinh^{-1} x = \ln [x + \sqrt{x^2 + 1}] \quad (\text{Ex 2.7.6 Q2})$$

Let $y = \sinh x$ and solve for x to find the inverse.

$$\begin{aligned} y &= \sinh x \\ &= \frac{1}{2} (e^x - e^{-x}) \end{aligned}$$

$$2y = e^x - e^{-x}$$

$$2ye^x = e^{2x} - 1$$

$$e^{2x} - 2ye^x - 1 = 0$$

$$(e^x)^2 - 2ye^x - 1 = 0$$

Let $u = e^x$.

$$\therefore u^2 - 2yu - 1 = 0.$$

$$\begin{aligned}
\therefore u &= \frac{-(-2y) \pm \sqrt{(-2y)^2 - 4(1)(-1)}}{2(1)} \\
&= \frac{2y \pm \sqrt{4y^2 + 4}}{2} \\
&= \frac{2y \pm 2\sqrt{y^2 + 1}}{2} \\
&= y \pm \sqrt{y^2 + 1}
\end{aligned}$$

But $u = e^x$

$$\therefore e^x = y \pm \sqrt{y^2 + 1}$$

Now, $e^x > 0$ and $y - \sqrt{y^2 + 1} < 0$

$$\therefore e^x = y + \sqrt{y^2 + 1}$$

$$\begin{aligned}
\therefore x &= \ln \left(y + \sqrt{y^2 + 1} \right) \\
&= f^{-1}(y)
\end{aligned}$$

$$\therefore \sinh^{-1} x = \ln \left(x + \sqrt{x^2 + 1} \right)$$

2.7.5 Revision Questions

The following questions are about the key ideas in this section.

1. Sketch the following graphs: (a) $y = \sinh^{-1} x$ (b) $y = \cosh^{-1} x$ (c) $y = \tanh^{-1} x$.
2. What are the domain and range of:
(a) $y = \sinh^{-1} x$ (b) $y = \cosh^{-1} x$ (c) $y = \tanh^{-1} x$.
3. Why do we need to restrict the domain of \cosh^{-1} and \tanh^{-1} but not \sinh^{-1} ?
4. **Exercise 2.7.6 Questions 1–2.**