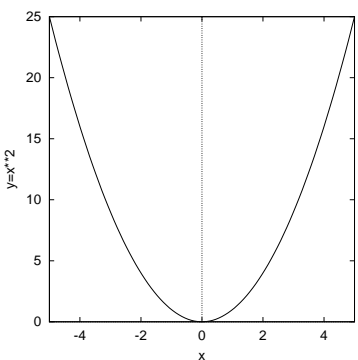


2.5 ONE-TO-ONE AND INVERSE FUNCTIONS

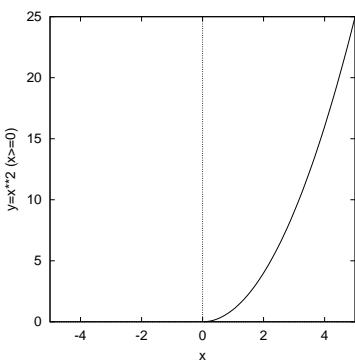
2.5.1 One-to one functions

Definition

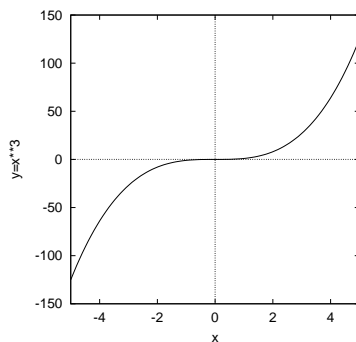
A function f is one-to-one _____ if its graph is cut only once by any horizontal line.
Consider the following graphs:



(a) $f(x)$



(b) $g(x)$



(c) $h(x)$

g and h are examples of one-to-one (often denoted 1-1) functions, f is not 1-1. The functions g and h have for each y value, only one x value.

The function f is a two-to-one function as each y -value has two x -values. e.g. if $y = a^2$, then $x = a$ or $x = -a$. An example of a many-to-one function is $f(x) = \sin x$ e.g. if $y = \frac{1}{2}$, then we have $x = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{13\pi}{6}, \dots$

2.5.2 Inverse Functions

The functions $f(x) = 2x$ and $g(x) = \frac{1}{2}x$ have the property that each is the inverse of the other.

$$f(g(x)) = f\left(\frac{1}{2}x\right) = 2\left(\frac{1}{2}x\right) = x$$

$$g(f(x)) = g(2x) = \frac{1}{2}2x = x$$

Similarly, the functions $f(x) = x^{1/3}$ and $g(x) = x^3$ are the reverse of each other.

$$f(g(x)) = f(x^3) = (x^3)^{1/3} = x$$

$$g(f(x)) = g(x^{1/3}) = (x^{1/3})^3 = x$$

We say that f is an inverse of g and g is an inverse of f .

We can also say that f and g are f and g are inverse functions.

Thus $f(x) = 2x$ and $g(x) = \frac{1}{2}x$ are inverse functions as are $f(x) = x^{1/3}$ and $g(x) = x^3$.

Notation

The *inverse of f* is commonly denoted as f^{-1} (read 'f inverse') thus

$$\underline{f(f^{-1}(x)) = f^{-1}(f(x)) = x}$$

So, we can write our inverse functions

$$f(x) = 2x \text{ and } g(x) = \frac{1}{2}x \text{ as}$$

$$f(x) = 2x \text{ and } \underline{f^{-1}(x) = \frac{1}{2}x} \text{ or}$$

$$g(x) = \frac{1}{2}x \text{ and } \underline{g^{-1}(x) = 2x}.$$

BEWARE: $f^{-1}(x)$ denotes the inverse of $f(x)$ *not* the reciprocal.

Question. Do all functions have inverses?

Definition A function f has an inverse if and only if it is one-to-one.

2.5.2.1 Exercises Do the following functions have inverses?

(a) $f(x) = x^2, x \leq 0$

(b) $g(x) = \sqrt{3x - 2}$

Hint: Sketch the functions.

2.5.3 Domain and Range of an Inverse

Consider the function $f(x) = 2x$, let $\text{Dom } f = \{x : x = 2, 4, 6, 8\}$ then

$$\text{Range } f = \{y : y = 4, 8, 12, 16\}.$$

Consider also the function $f^{-1}(x) = \frac{1}{2}x$ and let $\text{Dom } f^{-1} = \{x : x = 4, 8, 12, 16\}$ then $\text{Range } f^{-1} = \{y : y = 2, 4, 6, 8\}$.

If we compare we see that

$$\begin{aligned} \underline{\text{Dom } f^{-1}} &= \underline{\text{Range } f} \\ \underline{\text{Range } f^{-1}} &= \underline{\text{Dom } f} \end{aligned}$$

This is true for all f and f^{-1}

$$\begin{array}{l} \text{Dom } f^{-1} = \text{Range } f \\ \text{Range } f^{-1} = \text{Dom } f \end{array}$$

2.5.3.1 Exercises Find $\text{Dom } f^{-1}$ and $\text{Range } f^{-1}$ given f

(a) $f(x) = 2x - 4$

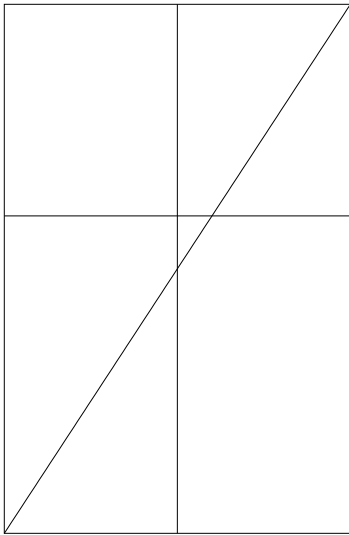
(b) $f(x) = x^2 - 1, x \geq 0$

(c) $f(x) = \sqrt{x - 3}, x > 3$

$$\begin{array}{l} \text{Dom } f^{-1} = \text{Range } f \\ \text{Range } f^{-1} = \text{Dom } f \end{array}$$

Find $\text{Dom } f^{-1}$ and $\text{Range } f^{-1}$:

(a) $f(x) = 2x - 4$. Draw the function.



$$\text{Dom } f(x) = x \in (-\infty, +\infty)$$

$$\text{Range } f(x) = y \in (-\infty, +\infty).$$

$$\text{Dom } f^{-1}(x) = x \in (-\infty, +\infty)$$

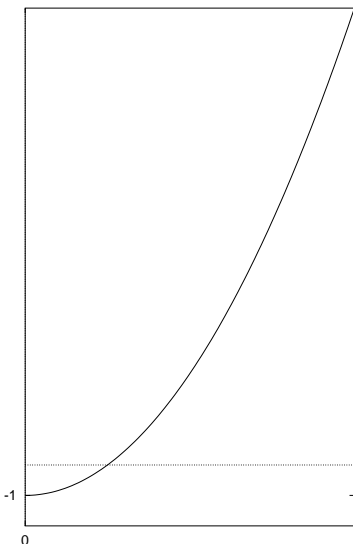
$$\text{Range } f^{-1}(x) = y \in (-\infty, +\infty)$$

Thus

$$\begin{aligned} \text{Dom } f^{-1} &= \text{Range } f \\ \text{Range } f^{-1} &= \text{Dom } f \end{aligned}$$

Find $\text{Dom } f^{-1}$ and $\text{Range } f^{-1}$

(b) $f(x) = x^2 - 1, x \geq 0$. Draw the function.



$$\text{Dom } f(x) = x \in [0, +\infty)$$

$$\text{Range } f(x) = y \in [-1, +\infty).$$

$$\text{Dom } f^{-1}(x) = x \in [-1, +\infty)$$

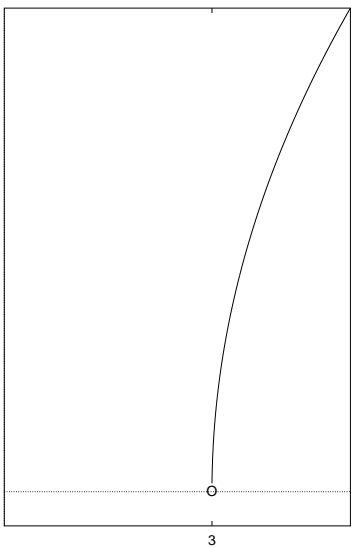
$$\text{Range } f^{-1}(x) = y \in [0, +\infty)$$

Thus

$\text{Dom } f^{-1} = \text{Range } f$ $\text{Range } f^{-1} = \text{Dom } f$

Find $\text{Dom } f^{-1}$ and $\text{Range } f^{-1}$ (c)

$f(x) = \sqrt{x-3}$, $x > 3$. Draw the function.



$$\text{Dom } f(x) = x \in (3, +\infty)$$

$$\text{Range } f(x) = y \in (0, +\infty).$$

$$\text{Dom } f^{-1}(x) = x \in (0, +\infty)$$

$$\text{Range } f^{-1}(x) = y \in (3, +\infty)$$

Thus

2.5.4 Finding $f^{-1}(x)$

If we let $y = f(x)$, we can find the inverse, f^{-1} , in terms of y , by solving for x .

$$\text{Let } y = f(x)$$

taking $\underline{f^{-1}}$ of both sides

$$f^{-1}(y) = \underline{f^{-1}(f(x))} = x$$

i.e. $x = f^{-1}(y)$ if and only if $y = f(x)$.

To find the inverse of the function $f(x)$.

1. Check f is 1-1. Why?
2. Let $y = f(x)$.
3. Solve for x to obtain $x = f^{-1}(y)$.
4. Write f^{-1} in terms of x .

2.5.4.1 Exercises

1. Find the inverse, if it exists, of each of the following:
 - (a) $f(x) = 2x + 4$
 - (b) $f(x) = 7x - 6$
 - (c) $f(x) = 3x^3 - 5$
 - (d) $f(x) = \frac{3}{x^2}$
 - (e) $f(x) = \frac{3}{x^2}, x < 0$

1. Check f is 1-1.
2. Let $y = f(x)$.
3. Solve for x to obtain $x = f^{-1}(y)$.
4. Write f^{-1} in terms of x .

(1a) Find the inverse, if it exists of:

$$f(x) = 2x + 4$$

$$y = 2x + 4$$

$$y - 4 = 2x$$

$$x = \frac{1}{2}(y - 4)$$

$$f^{-1}(x) = \frac{x - 4}{2}.$$

1. Check f is 1-1.
2. Let $y = f(x)$.
3. Solve for x to obtain $x = f^{-1}(y)$.
4. Write f^{-1} in terms of x .

(1b) Find the inverse, if it exists of:

$$f(x) = 7x - 6$$

$$y = 7x - 6$$

$$y + 6 = 7x$$

$$x = \frac{1}{7}(y + 6)$$

$$f^{-1}(x) = \frac{x + 6}{7}.$$

1. Check f is 1-1.
2. Let $y = f(x)$.
3. Solve for x to obtain $x = f^{-1}(y)$.
4. Write f^{-1} in terms of x .

(1c) Find the inverse, if it exists of:

$$f(x) = 3x^3 - 5$$

$$y = 3x^3 - 5$$

$$y + 5 = 3x^3$$

$$x^3 = \frac{y + 5}{3}$$

$$x = \left(\frac{y + 5}{3} \right)^{1/3}$$

$$f^{-1}(x) = \left(\frac{x + 5}{3} \right)^{1/3}$$

1. Check f is 1-1.

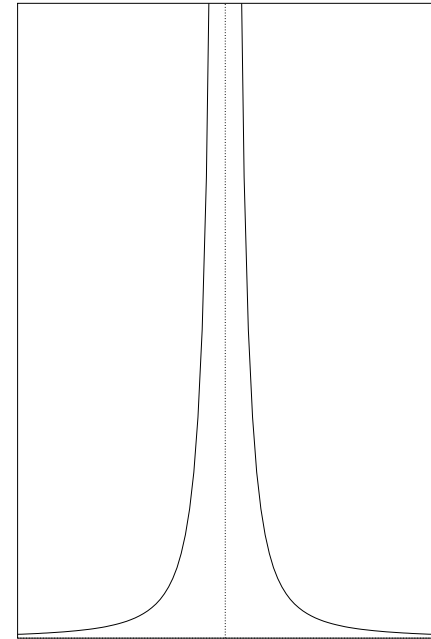
2. Let $y = f(x)$.

3. Solve for x to obtain $x = f^{-1}(y)$.

4. Write f^{-1} in terms of x .

(1d) Find the inverse, if it exists of: $f(x) = \frac{3}{x^2}$

The inverse does not exist because the function is not one-to-one.



1. Check f is 1-1.
2. Let $y = f(x)$.
3. Solve for x to obtain $x = f^{-1}(y)$.
4. Write f^{-1} in terms of x .

(1e) Find the inverse, if it exists of:

$$f(x) = \frac{3}{x^2}, \quad x < 0$$

$$y = \frac{3}{x^2}, \quad x < 0$$

$$x^2 = \frac{3}{y}, \quad x < 0$$

$$x = -\sqrt{\frac{3}{y}}$$

$$f^{-1}(x) = -\sqrt{\frac{3}{x}}$$

2. The temperature at which water boils will, to a point, decrease linearly as the altitude increases. Let the boiling point be a function of altitude such that $T_B = f(h)$, where T_B is the boiling point in $^{\circ}\text{C}$ and h is the altitude in metres. What is the meaning in practical terms of $f^{-1}(90)$? If $f^{-1}(90) = 3000$ evaluate $f^{-1}(85)$.

Hint. What is the boiling point of water at sea (ground) level ($h = 0$)?

We know that

$$T_B = A - Bh.$$

The boiling point of water at sea level is 100°C . Thus

$$100 = A.$$

The question tells us that

$$f^{-1}(90) = 3000.$$

Thus

$$\begin{aligned} 90 &= A - B(3000), \\ \Rightarrow 90 &= 100 - B(3000), \\ \Rightarrow B &= \frac{10}{3000} = \frac{1}{300}. \end{aligned}$$

Hence

$$T_B = 100 - \frac{1}{300}h$$

The solution $f^{-1}(85)$ is given by

$$\begin{aligned} 85 &= 100 - \frac{1}{300}h, \\ \Rightarrow h &= 15(300) = 4500. \end{aligned}$$

2.5.5 Revision of key ideas

The following questions are about the key ideas in this section.

1. What is meant by the expression that ' f is a 1-to-1 function'?
2. Suppose that f and g are functions. What is meant by the expression ' f is the inverse of g '?
3. What is the requirement for a function f to have an inverse?
4. How do the domain and range of a function f relate to the domain and range of its inverse function f^{-1} ?