2.4 THE HYPERBOLIC FUNCTIONS

Just as the trigonometric functions are based on the unit circle, the hyperbolic functions are based on the hyperbolas e^x and e^{-x}. The hyperbolic functions have many properties in common with the trigonometric functions. This similarity is reflected in the names of the hyperbolic functions.

The three elementary hyperbolic functions are the hyperbolic sine the hyperbolic cosine and the hyperbolic tangent. For convenience we shorten these names to

$\sinh x$ $\cosh x$ $\tanh x$

respectively, where the 'h' indicates the connection to the hyperbola.

Definition

$$\sinh x = \frac{e^x - e^{-x}}{2}$$

$$\cosh x = \frac{e^x + e^{-x}}{2}$$

$$\tanh x = \frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$

2.4.1 Graphs of Hyperbolic Functions

Cosh x

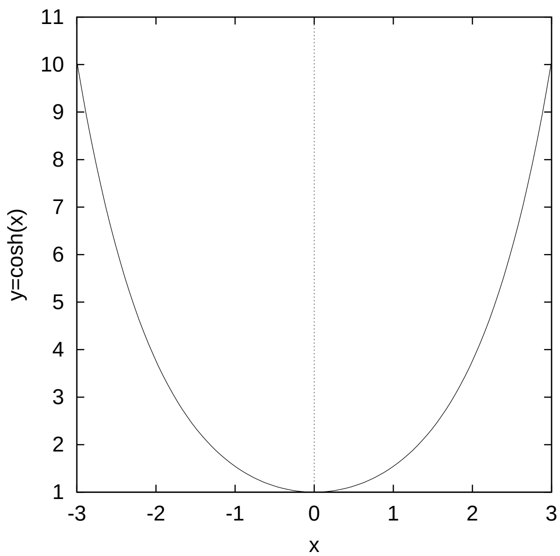

Figure 6: $y = \cosh x$

The graph of $\cosh x = \frac{1}{2}e^{x} + \frac{1}{2}e^{-x}$ can be obtained by separately graphing $\frac{1}{2}e^{x}$ and $\frac{1}{2}e^{-x}$ and then adding the y-coordinates together at each point.

We can see from the graph of $\cosh x$ the following important points:

1. $\cosh 0 = \underline{1}$
2. $\mathrm{Dom}\cosh x = \underline{\mathbb{R}}$
3. $\mathrm{Range}\cosh x = \underline{[1, \infty)}$
4. $\cosh x$ is an <u>even</u> function i.e. $\cosh(-x) = \underline{\cosh(x)}$.

Sinh x

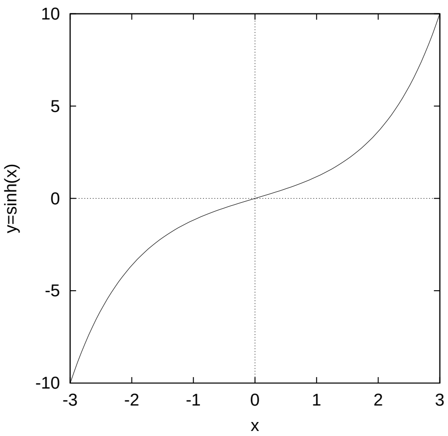

Figure 7: $y = \sinh x$

The graph of $\sinh x = \frac{1}{2}e^{x} - \frac{1}{2}e^{-x}$ be obtained in a similar manner to that of $\cosh x$.

Important points to note are:

1. $\sinh 0 = \underline{0}$
2. $\mathrm{Dom}\sinh x = \underline{\mathbb{R}}$
3. $\mathrm{Range}\sinh x = \underline{\mathbb{R}}$
4. $\sinh x$ is an <u>odd</u> function i.e. $\sinh(-x) = \underline{-\sinh x}$.

Tanh x

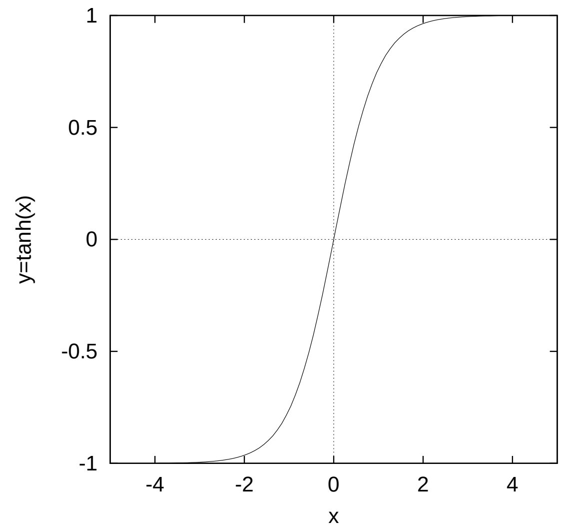

Figure 8: $y = \tanh x$

Important points to note are:

1. $\tanh 0 = \underline{0}$
2. $\mathrm{Dom}\tanh x = \underline{\mathbb{R}}$
3. $\mathrm{Range}\tanh x = \underline{(-1, 1)}$
4. $\tanh x$ is an <u>odd</u> function i.e. $\tanh(-x) = \underline{-\tanh x}$.

5. As $x \to \infty$ $\underline{\tanh x \to 1}$ and as $x \to -\infty$ $\underline{\tanh x \to -1}$

2.4.2 Reciprocal Hyperbolic Functions

The reciprocal hyperbolic functions are the hyperbolic secant
the hyperbolic cosecant and the hyperbolic cotangent.

Definition

$$\operatorname{sech} x = \frac{1}{\cosh x}$$
$$\operatorname{cosech} x = \frac{1}{\sinh x}$$
$$\coth x = \frac{1}{\tanh x}$$

The domain, range and graph of the above functions can be determined from the appropriate graph of $\sinh x$, $\cosh x$ or $\tanh x$. (See Exercise 2.4.5 Q3).

2.4.3 Hyperbolic Identities

The hyperbolic identities are very similar to the trigonometric identities.

$$\underline{\cosh^2 x - \sinh^2 x = 1}$$
$$\underline{1 - \tanh^2 x = \operatorname{sech}^2 x}$$
$$\underline{\coth^2 x - 1 = \operatorname{cosech}^2 x}$$

The first identity you will have to remember. The second and third can be found by dividing the first identity by $\underline{\cosh^2 x}$ and $\underline{\sinh^2 x}$ respectively.

$$\sinh(x \pm y) = \underline{\sinh x \cosh y \pm \cosh x \sinh y}$$
$$\cosh(x \pm y) = \underline{\cosh x \cosh y \pm \sinh x \sinh y}$$

2.4.3.1 Exercises

1. Show that $\cosh^2 x - \sinh^2 x = 1$
2. Prove that $\sinh(x + y) = \sinh x \cosh y + \cosh x \sinh y$

Note: 1 The result for $\cosh(x + y)$ is proved in a similar manner.

Note: 2 The results for $\cosh(x - y)$ and $\sinh(x - y)$ are found by letting $y = -y$ in $\cosh(x + y)$ and $\sinh(x + y)$ respectively.

1. Show that $\cosh^2 x - \sinh^2 x = 1$

Don't Panic!

$$\begin{aligned}
\cosh x &= \frac{1}{2}\left(e^{x} + e^{-x}\right), \\
\cosh^2 x &= \frac{1}{4}\left(e^{2x} + e^{-2x} + 2\right), \\
\sinh x &= \frac{1}{2}\left(e^{x} - e^{-x}\right), \\
\sinh^2 x &= \frac{1}{4}\left(e^{2x} + e^{-2x} - 2\right), \\
\cosh^2 - \sinh^2 x &= \frac{1}{4}\left(e^{2x} + e^{-2x} + 2\right) \\
&\quad - \frac{1}{4}\left(e^{2x} + e^{-2x} - 2\right) \\
&= \frac{1}{4} \times 4 \\
&= 1
\end{aligned}$$

2. Prove that

$\sinh(x+y) = \sinh x \cosh y + \cosh x \sinh y$

Hint

$$\cosh x + \sinh x = e^{x}$$

$$\cosh x - \sinh x = e^{-x}$$

$$\sinh(x+y) = \frac{e^{(x+y)} - e^{-(x+y)}}{2}$$

$$= \frac{e^{x} \cdot e^{y} - e^{-x} \cdot e^{-y}}{2}$$

$$= \frac{1}{2}\left[(\cosh x + \sinh x)(\cosh y + \sinh y) - (\cosh x - \sinh x)(\cosh y - \sinh y)\right]$$

$$= \sinh x \cosh y + \cosh x \sinh y$$

The following identities can be found by letting $y = x$ in $\cosh(x + y)$ or $\sinh(x + y)$. (See Exercise 2.4.5 Q4.)

$$\sinh 2x = 2 \sinh x \cosh x$$

$$\cosh 2x = \cosh^2 x + \sinh^2 x$$

$$\cosh^2 x = \frac{\cosh(2x) + 1}{2}$$

$$\sinh^2 x = \frac{\cosh(2x) - 1}{2}$$

2.4.4 Derivatives

The derivatives of the hyperbolic functions are given in figure 2.

$f(x)$	$f'(x)$	$f(x)$	$f'(x)$
$\sinh x$	$\underline{\cosh x}$	$\tanh x$	$\underline{\operatorname{sech}^2 x}$
$\cosh x$	$\underline{\sinh x}$	$\coth x$	$\underline{-\operatorname{cosech}^2 x}$

Table 2: Derivatives of hyperbolic functions

2.4.5 Exercises

1. Prove that $\frac{\mathrm{d}}{\mathrm{d}x}(\cosh x) = \sinh x$
2. Show that
   $\ln\left[\sinh x + \sqrt{\sinh^2 x + 1}\right] = x$
3. See exercise book.
4. See exercise book.
5. See exercise book.
6. See exercise book.
7. Prove the results in the table of derivatives of the hyperbolic function.

1. Prove that $\frac{\mathrm{d}}{\mathrm{d}x}(\cosh x) = \sinh x$

Don't Panic

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}\cosh x &= \frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{1}{2}\left(e^{x}+e^{-x}\right)\right],\\
&= \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}x}\left(e^{x}+e^{-x}\right),\\
&= \frac{1}{2}\left(e^{x}-e^{-x}\right),\\
&= \sinh x.
\end{aligned}
$$

2. Show that

$$\ln\left[\sinh x + \sqrt{\sinh^2 x + 1}\right] = x$$

Don't Panic

$$\cosh^2 x - \sinh^2 x = 1,$$

$$\sinh^2 x + 1 = \cosh^2 x,$$

$$\sqrt{\sinh^2 x + 1} = \sqrt{\cosh^2 x} = \cosh x.$$

$$\text{Let } y = \ln\left[\sinh x + \sqrt{\sinh^2 x + 1}\right],$$

$$\text{Then } y = \ln\left[\sinh x + \cosh x\right].$$

$$\text{Now } \cosh x = \frac{1}{2}\left(e^x + e^{-x}\right),$$

$$\sinh x = \frac{1}{2}\left(e^x - e^{-x}\right),$$

Hence $\cosh x + \sinh x = e^x$.

Thus $y = \ln\left[\sinh x + \cosh x\right] = \ln e^x = x$.

7. Prove that

(a) $\dfrac{\mathrm{d}\sinh x}{\mathrm{d}x} = \cosh x.$

(b) $\dfrac{\mathrm{d}\cosh x}{\mathrm{d}x} = \sinh x.$ (Already done, see question 2).

(c) $\dfrac{\mathrm{d}\tanh x}{\mathrm{d}x} = \operatorname{sech}^2 x.$

(d) $\dfrac{\mathrm{d}\coth x}{\mathrm{d}x} = -\operatorname{cosech}^2 x.$

$$\cosh x = \frac{1}{2}\left(e^{x} + e^{-x}\right),$$
$$\sinh x = \frac{1}{2}\left(e^{x} - e^{-x}\right).$$