

# DIFFERENTIATION

## 2.1 FUNCTIONS

### 2.1.1 The Basics

*Definition:* A function  $f$  is a rule which, for each number  $x$  in some subset  $A \in \mathbb{R}$ , assigns a (unique) number  $y = f(x) \in \mathbb{R}$ .

*Definition:* The set  $A$  is the set of all possible  $x$ -values of the function  $f$  and is called the domain of  $f$ . The domain of  $f$  is denoted by  $\text{Dom } f$ .

*Definition:* The set of all possible  $y$ -values of the function  $f$  is called the range of  $f$ . The range of  $f$  is denoted by  $\text{Range } f$ .

## Examples

1. Given  $f(x) = x^2$ .

Dom  $f = \mathbb{R}$  or  $(-\infty, \infty)$  or  
 $\{x : x \in \mathbb{R}\}$

Range  $f = \mathbb{R}^+$  or  $[0, \infty)$  or  
 $\{y : y \geq 0\}$

2. Given  $g(x) = x^2$ .  $x \geq 0$

Dom  $g = \mathbb{R}^+$

Range  $g = \mathbb{R}^+$

3. Given  $h(x) = \frac{1}{x-3}$ .

Dom  $h = \{x : x \neq 3\}$  or  
 $(-\infty, 3) \cup (3, \infty)$  or  $\mathbb{R} - \{3\}$ .

Range  $h = \{y : y \neq 0\}$  or  
 $(-\infty, 0) \cup (0, \infty)$  or  $\mathbb{R} - \{0\}$ .

Note that for the functions  $f$  and  $h$ , we choose the domain to be the largest set of  $x$  for which the function 'makes sense'. This is the natural domain of  $f$  and  $h$ .

*Definition:* Two functions  $f$  and  $g$  are equal if and only if  $f(x) = g(x)$  and  $\text{Dom } f = \text{Dom } g$ .

For example, consider the two functions  $f(x) = \sqrt{x^2 - 1}$  and  $g(x) = \sqrt{x^2 - 1}$   $x \geq 1$ .

The (natural) domain of  $f$  is  $x^2 - 1 \geq 0$  i.e.  $x^2 \geq 1 \Rightarrow \underline{|x| \geq 1}$ . i.e.

$\text{Dom } f = \{x : x \leq -1 \text{ or } x \geq 1\}$ .

The domain of  $g$  is given as

$\text{Dom } g = \underline{x \geq 1}$ .

Therefore  $\text{Dom } f \neq \text{Dom } g$  and so the two functions  $f$  and  $g$  are not equal.

*Definition:* Given a function  $y = f(x)$ ,  $\underline{x}$  is called the independent variable since it may be chosen freely from  $\text{Dom } f$  and  $\underline{y}$  is called the dependent variable since its value depends on the value chosen for  $x$ .

## 2.1.2 Functions defined Piecewise

Sometimes a function may be defined by formulas that have been ‘pierced together’.

The absolute value function,  $f(x) = |x|$  can be written in an equivalent piecewise form as

$$f(x) = \begin{cases} x, & \underline{x \geq 0} \\ -x, & \underline{x < 0} \end{cases}$$

### 2.1.3 Graph of a Function

*Definition:* The graph in the  $xy$ -plane ( $\mathbb{R}^2$ ) of a function  $f$  is defined to be the equation of the graph  $y = f(x)$ .

**Examples** Sketch the graphs of the following functions.

1.  $f(x) = x + 2$ .

2.  $g(x) = \begin{cases} x + 2, & x \neq 2 \\ 6, & x = 2. \end{cases}$

3.  $h(x) = \frac{x^2 - 4}{x - 2}$ .



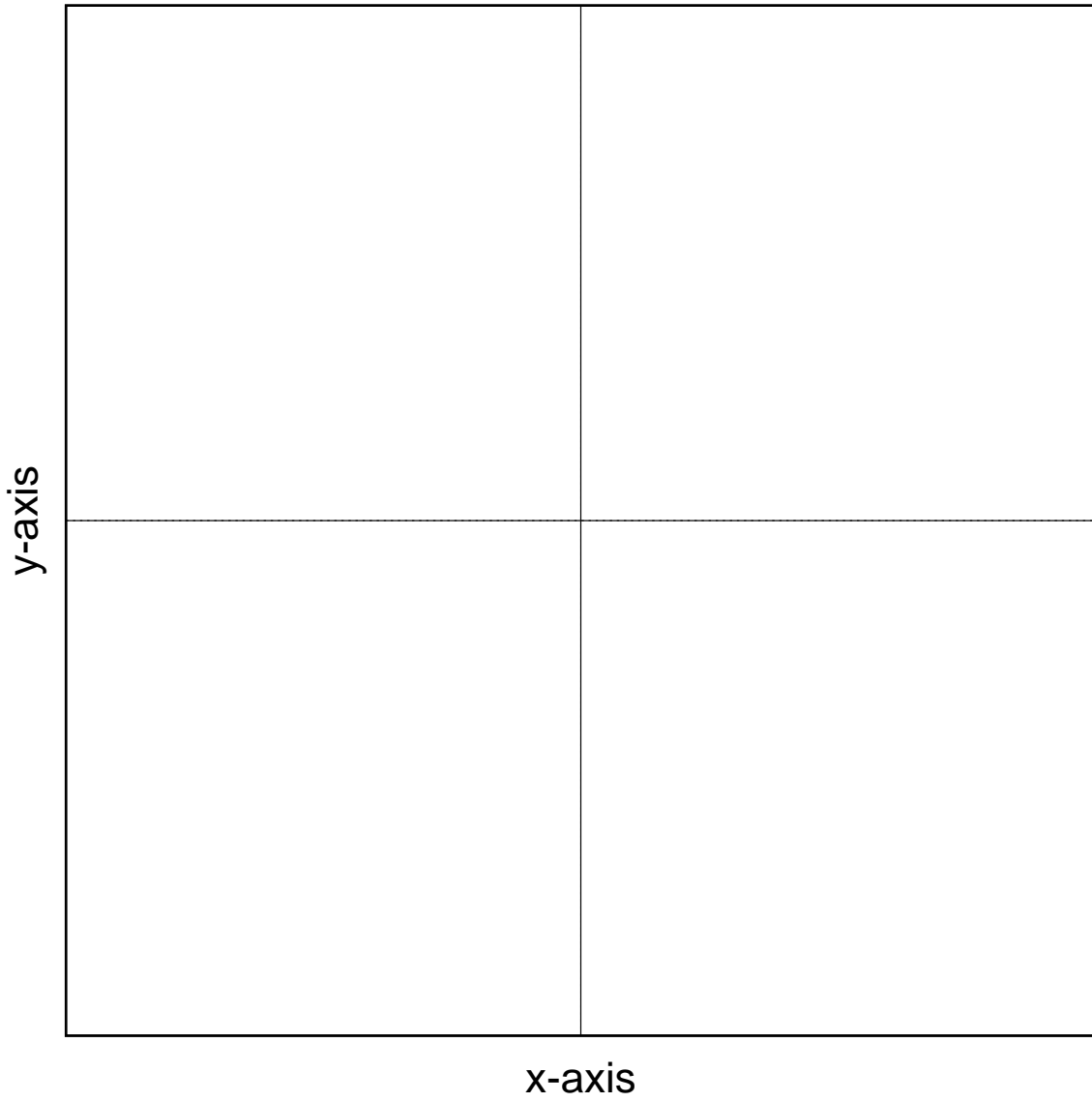


Figure 1: (a)  $f(x) = x + 2$

$$g(x) = \begin{cases} x + 2, & x \neq 2 \\ 6, & x = 2. \end{cases}$$

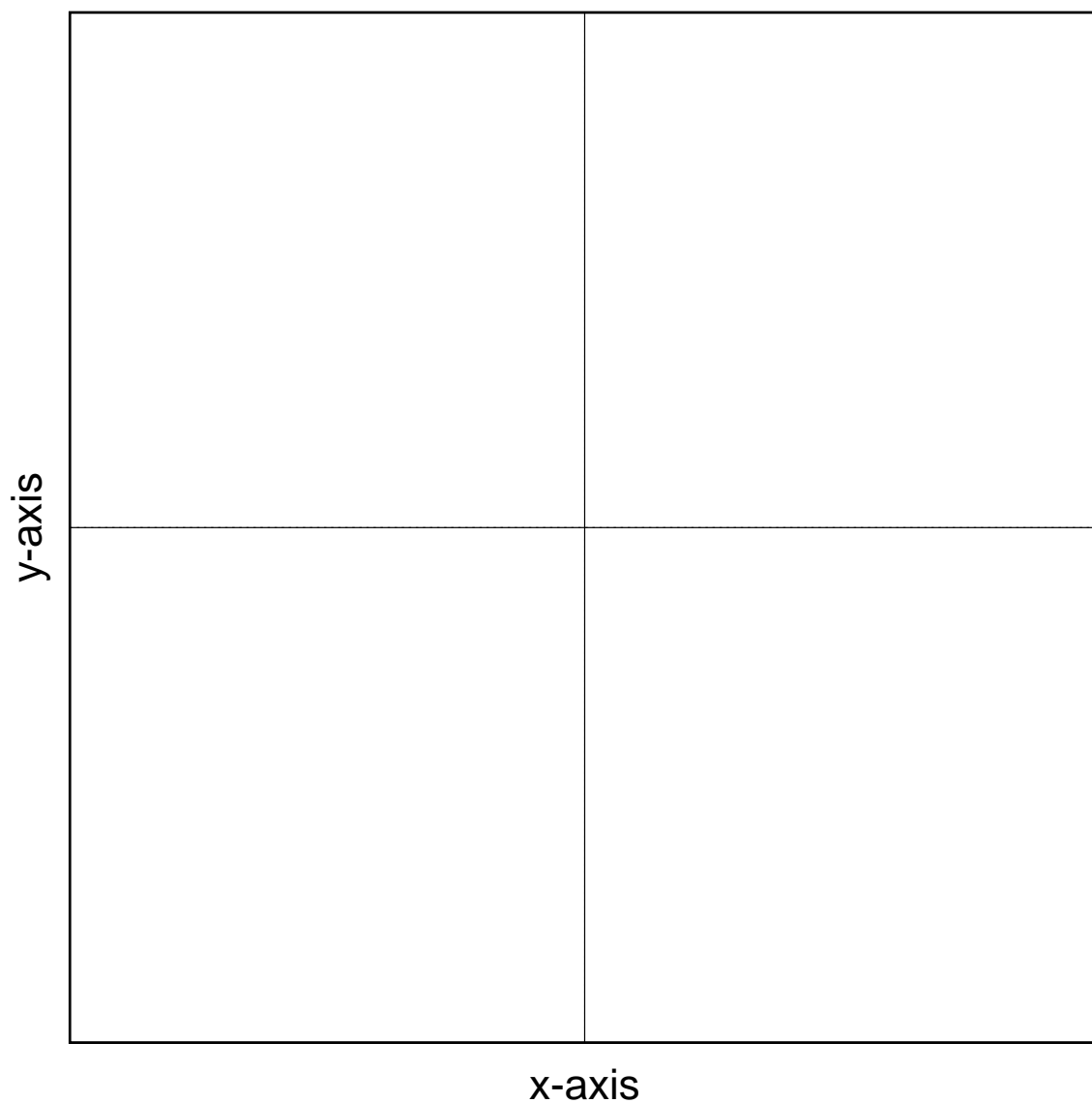


Figure 1: (b)  $g(x)$

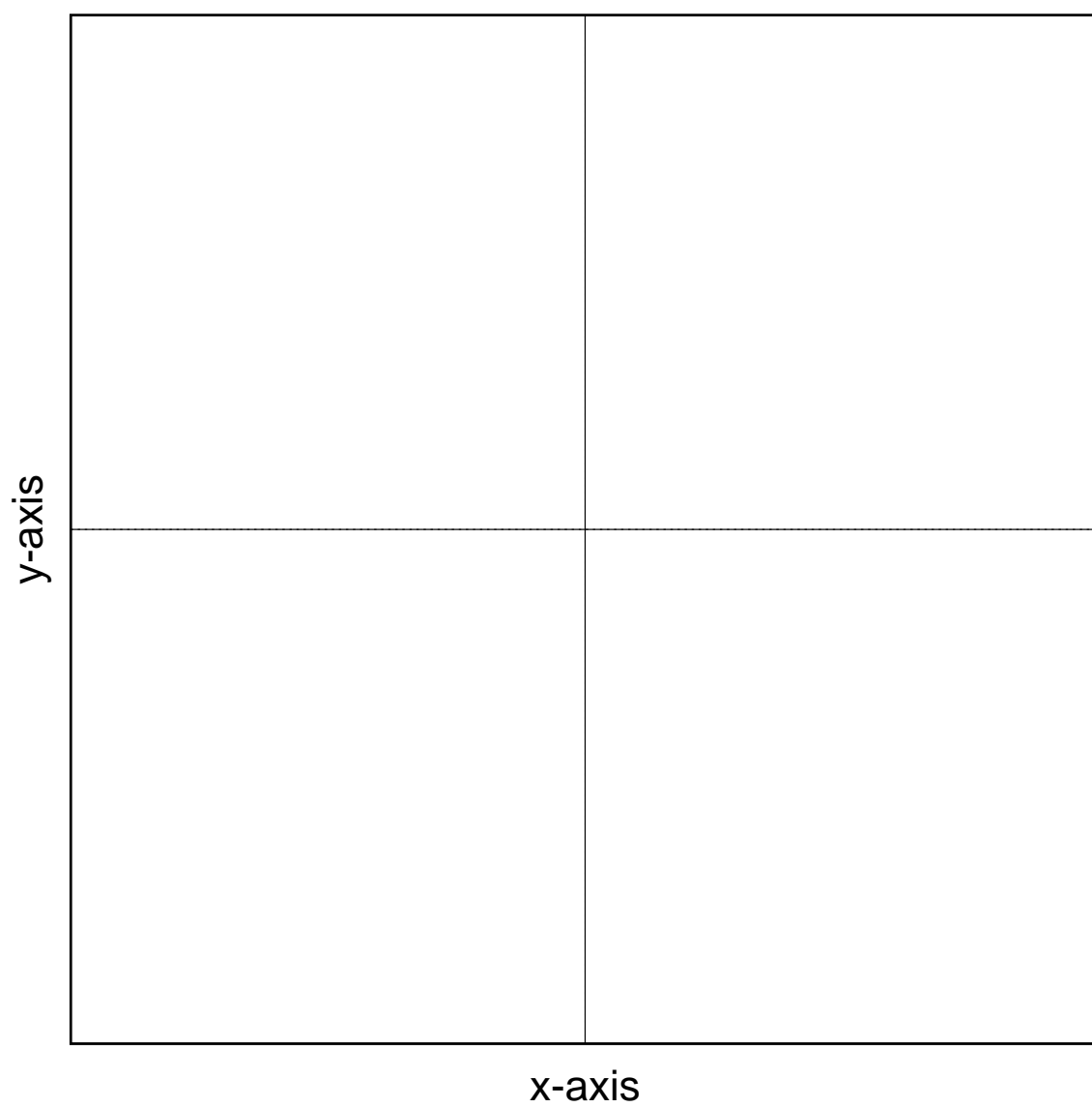


Figure 1: (c)  $h(x) = \frac{x^2 - 4}{x - 2}$

Important points to consider when sketching the graph of  $y = f(x)$  are:

1. Find  $\text{Dom } f$  (in particular, note where there are "gaps" in  $\text{Dom } f$ , and observe how  $y = f(x)$  behaves near the endpoints of the gaps),
2. Find the obvious zeroes (if any) of the function, and
3. Determine how  $f$  behaves as  $x$  becomes large and positive, and as  $x$  becomes large in absolute value, but negative.

In addition, if the graph of  $y = f(x)$  is known, then the graphs of each of the functions  $y = f(x - a)$ ,  $y = f(x) + a$  and  $y = \frac{1}{f(x)}$  are easy to determine:

1. The graph of  $y = f(x - a)$  has the same shape as  $y = f(x)$ , but is translated  $|a|$  units to the right (for  $a > 0$ ), or to the left (for  $a < 0$ ).
2. The graph of  $y = f(x) + a$  has the same shape as  $y = f(x)$ , but is translated  $|a|$  units up (for  $a > 0$ ), or down (for  $a < 0$ ).
3. The graph of  $y = \frac{1}{f(x)}$  has the same algebraic sign as  $f(x)$ , but is undefined when  $f(x) = 0$ , and approaches 0 whenever  $|f(x)|$  becomes large.

## Examples

1. Sketch  $y = |x - 3| + 2$ .

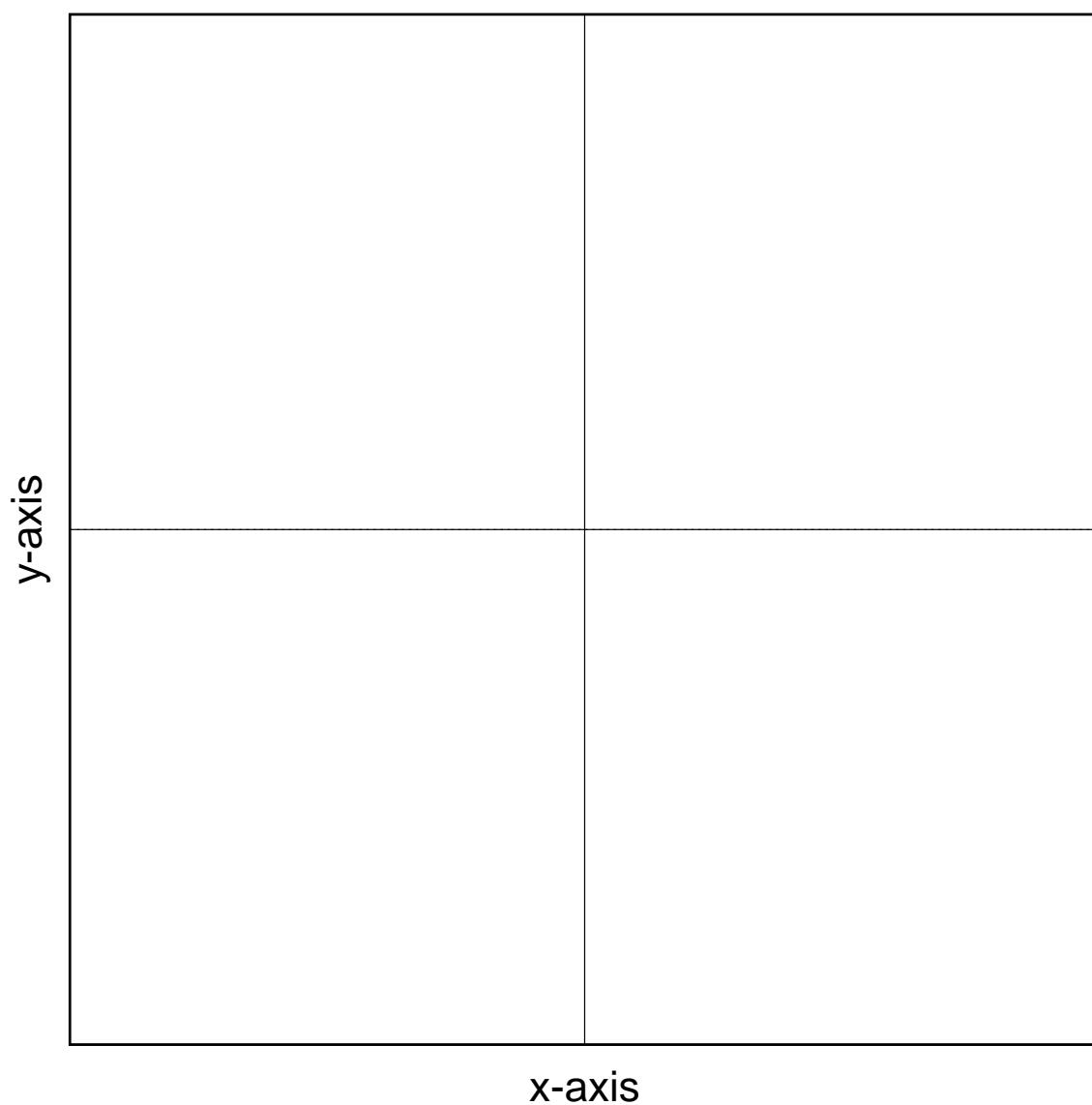


Figure 2: (a)

2. To find the domain and range of a function, it can sometimes be easier to sketch the function and then read off the domain and range.

Sketch the graph of  $f(x) = \sqrt{x - 2}$  and hence determine  $\text{Dom } f$  and  $\text{Range } f$ .

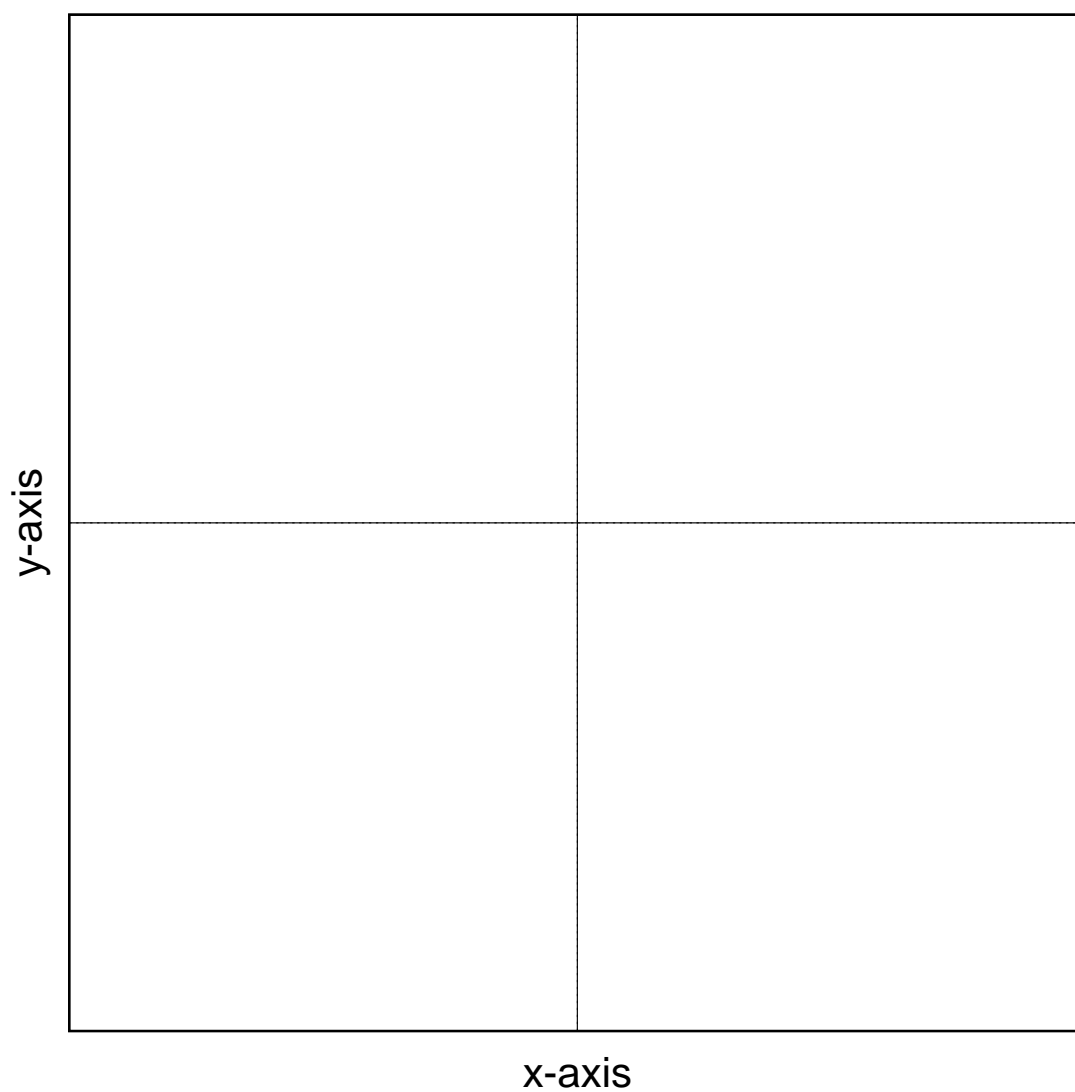


Figure 3: (b)  $f(x) = \sqrt{x - 2}$

### 2.1.4 Exercises

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## 2.1.5 Operations with Functions

If  $f$  and  $g$  are functions, and  $a \in \mathbb{R}$ , then

1.  $(f \pm g)(x) = \underline{f(x) \pm g(x)}$ , for  $x$  in both  $\text{Dom } g$  and  $\text{Dom } f$ .
2.  $(fg)(x) = \underline{f(x)g(x)}$ , for  $x$  in both  $\text{Dom } g$  and  $\text{Dom } f$ .
3.  $\frac{f}{g}(x) = \frac{f(x)}{\underline{g(x)}}$  for  $x$  in both  $\text{Dom } f$  and  $\text{Dom } g$ , and  $g(x) \neq 0$ .
4.  $(af)(x) = \underline{af(x)}$ , for  $x \in \text{Dom } f$ .

## Examples

1. Let  $f(x) = 1 + \sqrt{x - 2}$  and  $g(x) = x - 1$ . Find  $(f + g)(x)$  and its domain.
2. Let  $f(x) = \sqrt{2 - x}$  and  $g(x) = \sqrt{1 + x}$ . Find  $(f - g)(x)$  and its domain.
3. Let  $f(x) = \sqrt{x}$  and  $g(x) = 3\sqrt{x}$ . Find  $(fg)(x)$  and its domain.
4. Let  $f(x) = \sqrt{x - 5}$  and  $g(x) = x + 3$ . Find  $(f/g)$  and its domain.

(1) Let  $f(x) = 1 + \sqrt{x - 2}$  and  $g(x) = x - 1$ .

Find  $(f + g)(x)$  and its domain.

$$(f \pm g)(x) = f(x) \pm g(x), \text{ for } x \text{ in both Dom } g \text{ and Dom } f.$$

$$\text{Dom } f = \{x : x \geq 2\}$$

$$\text{Dom } g = \{x : x \in \mathfrak{R}\}$$

$$\begin{aligned}\text{Dom } f + g &= \{x : x \geq 2\} \cap \{x : x \in \mathfrak{R}\} \\ &= \{x : x \geq 2\}\end{aligned}$$

$$(f + g)(x) = \sqrt{x - 2} + x \text{ with}$$

$$\text{Dom}(f + g)(x) = \{x : x \geq 2\}$$

(2) Let  $f(x) = \sqrt{2 - x}$  and  
 $g(x) = \sqrt{1 + x}$ .

Find  $(f - g)(x)$  and its domain.

$$(f \pm g)(x) = f(x) \pm g(x), \text{ for } x \text{ in both Dom } g \text{ and Dom } f.$$

$$\text{Dom } f = \{x : x \leq 2\}$$

$$\text{Dom } g = \{x : x \geq -1\}$$

$$\begin{aligned}\text{Dom } f - g &= \{x : x \leq 2\} \cap \{x : x \geq -1\} \\ &= \{x : -1 \leq x \leq 2\}\end{aligned}$$

$$(f - g)(x) = \sqrt{x - 2} - \sqrt{1 + x} \text{ with}$$

$$\text{Dom}(f - g)(x) = \{x : -1 \leq x \leq 2\}$$

(3) Let  $f(x) = \sqrt{x}$  and  $g(x) = 3\sqrt{x}$ .

Find  $(fg)(x)$  and its domain.

$$(fg)(x) = f(x)g(x), \text{ for } x \text{ in both} \\ \text{Dom } g \text{ and Dom } f.$$

**Note:**  $\text{Dom } (fg) \neq \text{Dom } (3x) = \mathbb{R}$

$$\text{Dom } f = \{x : x \geq 0\}$$

$$\text{Dom } g = \{x : x \geq 0\}$$

$$\begin{aligned}\text{Dom } (fg)(x) &= \{x : x \geq 0\} \cap \{x : x \geq 0\} \\ &= \{x : x \geq 0\}\end{aligned}$$

$$(fg)(x) = 3x \text{ with}$$

$$\text{Dom}(fg)(x) = \{x : x \geq 0\}$$



(4) Let  $f(x) = \sqrt{x - 5}$  and  $g(x) = x - 7$ .  
Find  $f/g(x)$  and its domain.

$$\frac{f}{g}(x) = \frac{f(x)}{g(x)} \text{ for } x \text{ in both Dom } f \text{ and Dom } g, \text{ and } g(x) \neq 0.$$

$$\text{Dom } f = \{x : x \geq 5\}$$

$$\text{Dom } g = \{x : x \in \mathfrak{R}\}$$

$$\{x : g(x) = 0\} = \{x = 7\}$$

$$\begin{aligned}\text{Dom } \frac{f}{g} &= \{x : x \geq 5\} \cap \{x : x \in \mathfrak{R}\} \\ &\quad - \{x = 7\} \\ &= \{x : x \in [5, 7) \cup (7, \infty)\}\end{aligned}$$

$$(f/g)(x) = \frac{\sqrt{x-5}}{x-7} \text{ with}$$

$$\text{Dom}(f/g)(x) = \{x : x \in [5, 7) \cup (7, \infty)\}$$

## Composition of Functions.

*Definition:* Given functions  $f$  and  $g$ , the composition of  $f$  with  $g$ , denoted by  $\underline{f \circ g}$ , is the function defined by  $\underline{(f \circ g)(x) = f(g(x))}$ .

The domain of the composition of  $f$  with  $g$  is given by

$$\underline{\text{Dom}(f \circ g) = \{x \in \text{Dom } g \text{ and } g(x) \in \text{Dom } f\}}.$$

1. Note:  $f \circ g$  is sometimes described as 'a function of a function'.

The domain of the composition of  $f$  with  $g$  is given by

$$\underline{\text{Dom}(f \circ g) = \{x \in \text{Dom } g \text{ and } g(x) \in \text{Dom } f\}}.$$

2. Note: The domain may seem complicated but it is only sensible. To evaluate  $f(g(x))$ ,  $x$  must be in the domain of  $g$  to evaluate to evaluate  $g(x)$ , and also  $g(x)$  must be in the domain of  $f$  to evaluate  $f(g(x))$ .

## Examples

1. Find (i)  $f \circ g$  and (ii)  $g \circ f$  for each of the following.

(a)  $f(x) = 2x, g(x) = x^2 + 1.$

(b)  $f(x) = 3x - 2, g(x) = |x|.$

(c)  $f(x) = \sqrt{x+1}, g(x) = x - 2.$

2. Find  $(f \circ g)(x)$  if  $f(x) = x^2 + 3$  and  $g(x) = x - 1$ . State the domain of  $f \circ g$ .

3. Find  $(f \circ g)(x)$  and its domain if  $f(x) = \sqrt{x}$  and  $g(x) = x - 1$ .

4. Find  $(f \circ g)(x)$  if  $f(x) = x^2 + 3$  and  $g(x) = \sqrt{x}$ . State the domain of  $f \circ g$ .

5. Express  $h(x) = \sqrt{4 - 3x}$  as a composition of two functions  $f(x)$  and  $g(x)$  such that  $h(x) = f(g(x))$ .

## 2.1.6 Exercises

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## 2.1.7 Revision of key ideas

The following questions are about the key ideas in this section.

1. Suppose that  $y = f(x)$ . What it is meant by *the domain of the function  $f$*  and by *the range of the function  $f$* ?
2. Suppose that  $a = f(b)$ . Which is the *independent variable* and which is the *dependent variable*?

3. Given the graph  $y = f(x)$  explain how it is related to the graphs: (i)  $y = f(x - 1)$ , (ii)  $y = f(x) + a$  and (iii)  $y = 1/f(x)$ .
4. Given two functions  $f$  and  $g$  what do we mean by the *composition of  $f$  with  $g$* ? What is the definition of  $\text{Range } f \circ g$  and  $\text{Dom } f \circ g$ ?