2.3 ELEMENTARY DIFFERENTIATION

2.3.1 The basics

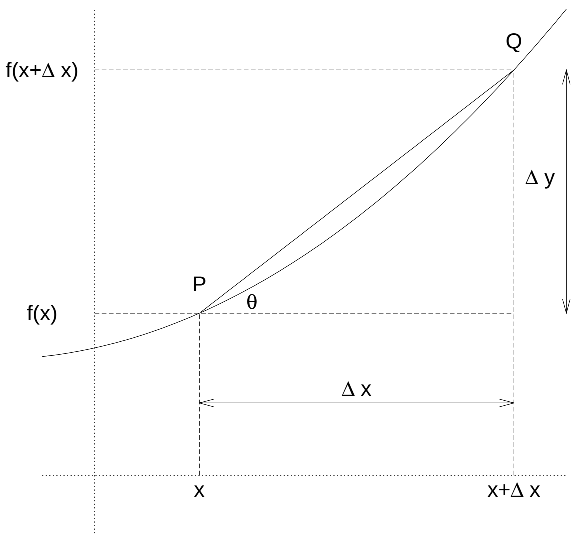

Figure 5:

Consider the diagram on the left where Δx and Δy are <u>small</u> <u>changes</u> in the values of x and y respectively.

$$\begin{aligned}\text{gradient of } PQ &= \tan\theta \\ &= \frac{\Delta y}{\Delta x} \\ &= \frac{f(x+\Delta x) - f(x)}{\Delta x}\end{aligned}$$

Looking at our diagram, if we let $\Delta x \to 0$ i.e. let Q approach P, then our line PQ becomes a tangent to the curve at P. The slope of the curve at P is the slope of the tangent to the curve at P and is given by

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} &= \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} \\ &= \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}\end{aligned}$$

The notation $f'(x)$ is also used to represent the gradient of the tangent to $y = f(x)$ at $(x, f(x))$ i.e.

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}.$$

We say that $f'(x)$ (or $\frac{\mathrm{d}y}{\mathrm{d}x}$) is the <u>derivative</u> of the function $y = f(x)$ at any point x. Other notations for the derivative include y' or $D_x f$. The operation of obtaining $f'(x)$ is called <u>differentiation</u>. Geometrically, $f'(x)$ represents the slope of the tangent line to the graph of $f(x)$ at the point $(x, f(x))$.

An alternative interpretation of $\frac{\mathrm{d}y}{\mathrm{d}x}$ is that it is the instantaneous rate of change of of y with respect to x'. We defined Δx and Δy as small changes in x and y respectively. Therefore, $\dfrac{\Delta y}{\Delta x}$ is the rate of change in y with respect to the change in x. If Δx and Δy are relatively small, then $\dfrac{\Delta y}{\Delta x} \approx \dfrac{\mathrm{d}y}{\mathrm{d}x}$.

2.3.2 Some Differentiable Functions

The functions and their derivatives in table 1 should be known to you.

$f(x)$	$f'(x)$	$f(x)$	$f'(x)$
c	$\underline{0}$	$\tan x$	$\underline{\sec^2 x}$
x^n	$\underline{nx^{n-1}}$	e^x	$\underline{e^x}$
$\sin x$	$\underline{\cos x}$	$\ln x$	$\underline{\frac{1}{x}}$
$\cos x$	$\underline{-\sin x}$		

Table 1: Standard differentiable functions

2.3.3 Rules for Finding Derivatives

If f and g are two functions which are differentiable at x, and $\alpha \in \mathbb{R}$, then $f+g$, αf, fg and f/g (if $g(x) \neq 0$), are differentiable at x and we have:

1. $(f+g)'(x) = \underline{f'(x) + g'(x)}$

2. $(\alpha f)'(x) = \underline{\alpha f'(x)}$

3. $(fg)'(x) = \underline{f(x)g'(x) + f'(x)g(x)}$
   [the **product rule**]

4. $\left(\dfrac{f}{g}\right)'(x) = \underline{\dfrac{g(x)f'(x) - f(x)g'(x)}{[g(x)]^2}}$
   [the **quotient rule**]

5. If $y = f(g(x))$ then
$\frac{dy}{dx} = \underline{f'(g(x)) \cdot g'(x)}$ or equivalently
$\frac{dy}{dx} = \underline{\frac{dy}{du}\frac{du}{dx}}$ where $y = \underline{f(u)}$ with
$u = \underline{g(x)}$.
[the **chain rule** or function of a function rule]

2.3.4 Exercises

Find the derivatives of the following functions:

(a) $y = 3x^4 - 7x^3 + 4x^2 + 3x - 4$

(b) $y = 2x^5 \cos x$

(c) $y = \cot x$

(d) $y = \sin^3 x$

Find the derivatives of the function:

(a) $$y = 3x^4 - 7x^3 + 4x^2 + 3x - 4$$

We will use the result that
$(x^n)' = n\left(x^{n-1}\right)$.

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{\mathrm{d}}{\mathrm{d}x}3x^4 - \frac{\mathrm{d}}{\mathrm{d}x}7x^3 + \frac{\mathrm{d}}{\mathrm{d}x}4x^2 + \frac{\mathrm{d}}{\mathrm{d}x}3x \\ &\quad - \frac{\mathrm{d}}{\mathrm{d}x}4 \\ &= 12x^3 - 21x^2 + 8x + 3.\end{aligned}$$

Find the derivatives of the function:

(b) $$y = 2x^5 \cos x$$

We need to use the **product rule**
$(fg)'(x) = f(x)g'(x) + f'(x)g(x)$

$$\begin{aligned} f &= 2x^5 \\ f' &= 10x^4 \\ g &= \cos x \\ g' &= -\sin x. \\ \frac{\mathrm{d}y}{\mathrm{d}x} &= -2x^5 \sin x + 10x^4 \cos x \end{aligned}$$

Find the derivatives of the function:

(c) $$y = \cot x$$

We need to write

$$
\begin{aligned}
y &= \cot x \\
&= \frac{1}{\tan x}
\end{aligned}
$$

and then use the **quotient rule**

$$
\left(\frac{f}{g}\right)'(x) = \frac{g(x)f'(x) - f(x)g'(x)}{[g(x)]^2}
$$

Here we have

$$
\begin{aligned}
f &= 1, & f' &= 0, \\
g &= \tan x, & g' &= \sec^2 x,
\end{aligned}
$$

Thus

$$
\begin{aligned}
\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{(\tan x)\cdot 0 - 1\cdot \sec^2 x}{\tan^2 x}, \\
&= -\frac{\sec^2 x}{\tan^2 x}
\end{aligned}
$$

Find the derivatives of the function:

(d) $$y = \sin^3 x$$

We need to use the **chain rule**. Put $u = \sin x$ then

$$\begin{aligned} y &= u^3, & u &= \sin x, \\ \frac{\mathrm{d}y}{\mathrm{d}u} &= 3u^2, & \frac{\mathrm{d}u}{\mathrm{d}x} &= \cos x, \\ \frac{\mathrm{d}y}{\mathrm{d}u} &= 3\sin^2 x & \frac{\mathrm{d}u}{\mathrm{d}x} &= \cos x, \\ \frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{\mathrm{d}y}{\mathrm{d}u} \cdot \frac{\mathrm{d}u}{\mathrm{d}x}, \\ \frac{\mathrm{d}y}{\mathrm{d}x} &= 3\sin^2 x \cdot \cos x \end{aligned}$$

2.3.5 Higher Order Derivatives

Let f be a differentiable function and f' be the first derivative of f. If f' is itself differentiable, then the second derivative of f is $(f')' = f''$.

For example, if $f(x) = x^3 - x^2 - x + 1$, then

$$f'(x) = \underline{\hspace{4cm}} \quad \text{and}$$
$$f''(x) = \underline{\hspace{2cm}}$$

$$3x^2 - 2x - 1$$

$$6x - 2$$

Similarly, if f'' is differentiable then the third derivative of f is f'''. Alternative notations are as follows:

$$f' = f^{(1)}, \qquad f'' = f^{(2)},$$
$$f''' = f^{(3)}, \qquad f'''' = f^{(4)}, \text{ etc.}$$

Note that if f is a function and n a positive integer, then the n^{th}-order derivative of f, if it exists, is $f^{(n)} = \left[f^{(n-1)}\right]'$.

For example, if

$$f(x) = x^{1/2},$$

$$f'(x) = \underline{\qquad\qquad}$$

$$f''(x) = \underline{\qquad\qquad\quad}$$

$$f'''(x) = \underline{\qquad\qquad}$$

$$f^{(4)}(x) = \left[f^{(3)}(x)\right]'$$

$$= \underline{\qquad\qquad\qquad}$$

$$\frac{1}{2}x^{-1/2}$$

$$-\frac{1}{4}x^{-3/2}$$

$$\frac{3}{8}x^{-5/2}$$

$$-\frac{15}{16}x^{-7/2}$$

Notations

1. If $y = f(x)$, and f' is used for the first derivative, then the notations f'' and y'' are used for the second-order derivative, and $f^{(n)}$ and $y^{(n)}$ are used for the nth-order derivative.
2. If $y = f(x)$, and the notation $\frac{\mathrm{d}y}{\mathrm{d}x}$ is used for the first derivative, then the second-order derivative is given by $\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right) = \frac{\mathrm{d}^2 y}{\mathrm{d}x^2}$, or $\frac{\mathrm{d}^2 f}{\mathrm{d}x^2}$. The nth-order derivative is then denoted by $\frac{\mathrm{d}^n y}{\mathrm{d}x^n}$ or $\frac{\mathrm{d}^n f}{\mathrm{d}x^n}$.
3. If $y = f(x)$, and the notation $D_x y$ is used, then the second derivative is denoted by $D_x^2 y$, or $D_x^2 f$. The nth-order derivative is then denoted by $D_x^n y$ or $D_x^n f$.

2.3.6 Differentiation and YOU!

Differentiation is an important tool for solving problems in commerce, economics, finance, engineering and science. (It also has applications in many other areas).

(a) What is your degree?

(b) Give an example of a problem relevant to your degree that can be solve using differentiation. You do not need to derive an equation or solve the problem. You should provide sufficient detail so that it is clear why differentiation is required to solve the problem.

(If you can not think of an example for your degree course then go and see your lecturer).

2.3.7 Revision of key ideas

The following questions are about the key ideas in this section.

1. Given $f(x)$ and $g(x)$ you will *need* to know how to:
   (a) Differentiate $f(x)g(x)$.
   (b) Differentiate $f(x)/g(x)$.
   (c) Differentiate $f(g(x))$.
2. You will need to learn the table of differentiable functions.