MATH111 – Spring 2007
Tutorial Sheet – Week 12

This tutorial sheet covers sections 13.5–13.7 of the notes ('constant effort harvesting').

Chapter 13. Harvesting

Revision of Key Ideas

The following questions are about the key ideas in sections 13.5–13.7.

1. The logistic differential equation is
$$x' = rx\left((1 - \frac{x}{K}\right).$$
Write down the logistic differential equation with constant effort harvesting.

2. Explain what the term $-Ex$ 'means' in the constant effort harvesting model.

3. For real fisheries is it always reasonable to assume that the number of fish caught per unit time is proportional to the effort expended in fishing?

4. The logistic differential equation with constant effort harvesting is
$$x' = rx\left((1 - \frac{x}{K}\right) - Ex.$$
   (a) Find the steady-states of this equation and determine their stability as a function of the parameter E.

   (b) Draw a steady-state diagram for this model showing how the steady-state solutions and their stability vary as a function of E. Discuss the implications of your figure.

   (c) Derive the equilibrium yield equation for the model and find the maximum sustainable yield.

5. Comment on the relative advantages and disadvantages of constant yield harvesting and constant effort harvesting.

Exercises

Question 3 is a straightforward regurgitation of material from the lecture notes.

1. A population of sandhill cranes (*Grus canadiensis*) has been modelled by a logistic equation with carrying capacity of 194,600 members and intrinsic growth rate 0.0987year^{-1}. What is the maximum yield that can be obtained from constant effort harvesting and what is the corresponding value for the effort? What is the equilibrium population size and the yield that results when the effort is 0.05 year^{-1}. You may quote appropriate results from your lecture notes.

2. Consider the Gompertz model with constant effort harvesting

$$\frac{\mathrm{d}x}{\mathrm{d}t} = r\log\left[\frac{K}{x}\right] - Ex. \tag{1}$$

(a) The change of base rule states that

$$\log_a z = \frac{\log_b z}{\log_b a}.$$

Use the change of base rule to write the model equation in the form

$$\frac{\mathrm{d}x}{\mathrm{d}t} = r^* \ln\left[\frac{K}{x}\right] - Ex, \tag{2}$$

where $r^* = r/\ln 10$.

(b) Find the steady-state solution of equation (**??**) and determine its stability as a function of the parameter E.

Note that $x = 0$ is not a steady-state solution of equation (**??**).

(c) (i) Write down the yield function for system (**??**).

(ii) Find the value of E that maximises the yield and the corresponding maximum value.

(iii) Sketch the yield as a function of the parameter E.

3. Consider the logistic equation with proportional harvesting

$$\frac{\mathrm{d}x}{\mathrm{d}t} = rx\left(1 - \frac{x}{K}\right) - px, \quad x(0) = x_0$$

where x is the population density of an animal in an environment, t is time, $r > 0$ is the static birth rate, $K > 0$ is the carrying capacity of the environment, $p \geq 0$ is the proportional harvesting parameter and $x_0 > 0$ is the initial population density.

(a) Consider the model without harvesting ($p = 0$).

(i) Sketch the function $y = rx\left(1 - \frac{x}{K}\right)$.

(ii) Describe the dynamic evolution of the population density for $x_0 < K$ and $x_0 > K$.

(b) Find the steady-state solutions of the logistic equation with proportional harvesting.

(c) Determine the stability of the steady-state solutions of the logistic equation with proportional harvesting as a function of the parameter p.

(d) Show that the value of p which maximises the equilibrium yield is $p = \frac{r}{2}$. What is the maximal equilibrium yield?