MATH111 – Spring 2007
Tutorial Sheet – Week 10

This tutorial sheet covers chapters 12 of the notes.

Chapter 12. Linear Stability Analysis of Steady-State Solutions

Revision of Key Ideas

The following questions are about the key ideas in this chapter.

1. What does it mean for a steady-state x^* to be called *trivial*?
2. Suppose that x^* is a steady-state solution of the differential equation

$$\frac{\mathrm{d}x}{\mathrm{d}t} = f(x) .$$

   (a) What is the condition for x^* to be *stable*?

   (b) What is the reason why it is important to find the stability of a steady-state?

3. (a) You should understand the derivation of the equation

$$\frac{\mathrm{d}\xi}{\mathrm{d}t} = \lambda\xi$$

   and its interpretation in terms of stability.

   (b) Suppose that x^* is a steady-state solution of the differential equation

$$\frac{\mathrm{d}x}{\mathrm{d}t} = f(x)$$

   with eigenvalue $\lambda < 0$. Why does this mean that the steady-state solution is stable?

4. (a) What information does a *response curve* contain?

   (b) Explain what you understand by the word '*bifurcation*'.

   (c) Suppose that x^* is a steady-state solution of the differential equation

$$\frac{\mathrm{d}x}{\mathrm{d}t} = f(x, r)$$

   Define what is meant by the phrases *limit-point bifurcation*, *pitchfork bifurcation* and *transcritical bifurcation.*

Exercises

1. The transmission dynamics of a disease in a population is represented by the equation

$$\frac{\mathrm{d}I}{\mathrm{d}t} = \beta I \left(1 - \frac{I}{K}\right) - \alpha I,$$

where I is the number of infected individuals in the population, β denotes the transmission coefficient, K is the total population size and α the recovery rate. Assume that $K > 0$, $\alpha > 0$ and $\beta > 0$.

   (a) Find the steady-state solutions of this model and determine their stability as a function of the recovery rate α).

   (b) Hence, or otherwise, determine the condition for the disease to be eradicated.

   (c) Draw a steady-state diagram for this model treating the recovery rate, α, as the control parameter. Indicate stable and unstable steady-state solutions using solid and dashed lines respectively.

2. Determine graphically the stability of the trivial steady-state solution for the differential equation

$$\frac{\mathrm{d}x}{\mathrm{d}t} = ax^3.$$

Consider the cases $a > 0$ and $a < 0$. What is the eigenvalue of the trivial steady-state solution?

3. (a) A population is governed by the differential equation

$$x' = x\left(e^{3-x} - 1\right).$$

   Find all steady-state solutions and determine their stability.

   (b) A fraction $p\,(0 < p < 1)$ of the population in part (a) is removed in unit time so that the population size is governed by the differential equation

$$x' = x\left(e^{3-x} - 1\right) - px.$$

   For what values of p is there a stable positive equilibrium?

4. Nisbet & Gurney (1983) suggested the following form for the per-capita growth rate

$$r(x) = r \exp\left[1 - \frac{x}{K}\right] - d$$

Consider the associated population model

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \left(r \exp\left[1 - \frac{x}{K}\right] - d\right) x, \quad x(0) = x_0,\ r > de^{-1}.$$

   (a) Find the steady-state(s) of the model. How do the number of steady-state solutions ($x^* \geq 0$) and their biological feasibility depend upon the values of r and d?

   (b) Explain why it is reasonable to assume that $r > de^{-1}$.

   (c) Sketch $\frac{\mathrm{d}x}{\mathrm{d}t}$ as a function of x. Hence determine how the long-term dynamics of the model depends upon the initial value x_0.