MATH111 – Spring 2006
Tutorial Sheet – Week 9

This tutorial sheet covers chapters 11 of the notes.

Chapter 11. First-Order Non-Linear Differential Equations

Revision of Key Ideas

The following questions are about the key ideas in this chapter.

1. Derive the linear population model, explaining all the inherent assumptions.
2. Explain the defects of the linear model.
3. Explain how graphical techniques can be used to explore the dynamic behaviour of the differential equation
$$\frac{\mathrm{d}x}{\mathrm{d}t} = f(x).$$
4. What does it mean for the point x^* to be a steady-state of the differential equation
$$\frac{\mathrm{d}x}{\mathrm{d}t} = f(x).$$
5. Consider the population model
$$\frac{\mathrm{d}x}{\mathrm{d}t} = f(x).$$
Explain why it is important to find the steady-states of the function $f(x)$.

Exercises

1. The population density of fish is modelled by the differential equation
$$\frac{\mathrm{d}u}{\mathrm{d}t} = f(u), \qquad u(t=0) = u_0,$$
where the function $f(u)$ has the following properties:
   - $f(0) = f(K_0) = f(K) = 0$ where $0 < K_0 < K$.
   - If $u \in (0, K_0)$ then $f(u) < 0$.
   - If $u \in (K_0, K)$ then $f(u) > 0$.
   - If $u > K$ then $f(u) < 0$.

   (a) Sketch the growth curve $f(u)$ as a function of u.

   (b) How does the long-term evolution of the differential equation depend upon the choice of the initial condition u_0?

   (c) A disease spreads through the population reducing the population to a density $K_0/2$. What happens to the population? Justify your answer.

2. The rate of removal of substrate in an anaerobic digester is given by

$$\frac{\mathrm{d}S}{\mathrm{d}t} = \frac{q}{V}(S_0 - S) - kS,$$

where: S is the concentration of volatile solids in the reactor (kg m^{-3}); S_0 is the concentration of volatile solids entering the reactor (kg m^{-3}); V is the volume of the reactor (m^3); k is the first order rate constant for the degradation of volatiles solids (day^{-1}). q is the volumetric flow-rate through the reactor (m^{3}day^{-1}); and t is time (day);

The production rate of methane inside the reactor (m^{3}day^{-1}) is given by

$$g = yB_0(S_0 - S)q$$

where: B_0 is a biodegradability factor; and y is the specific methane productivity (m^{3}kg VS destroyed^{-1}).

   (a) Determine the steady-state concentration of volatile solids in the reactor.
   (b) Sketch a figure showing how, under steady-state conditions, the production rate of methane inside the reactor (g) depends upon the flow rate (q).

3. For some organisms finding a suitable mate may cause difficulties at low population densities, and a more realistic equation for population growth under these conditions may be

$$\frac{\mathrm{d}N}{\mathrm{d}t} = rN^2, \quad r > 0, N(0) = N_0.$$

   (a) Solve this problem and show that the solution becomes infinite in finite time.
   (b) The model above may be improved to

$$\frac{\mathrm{d}N}{\mathrm{d}t} = rN^2\left(1 - \frac{N}{K}\right).$$

      (i) Sketch the growth curve $f(N)$ as a function of N.
      (ii) How does the long-term evolution of the differential equation depend upon the choice of the initial condition N_0?

4. Suppose a population satisfies a logistic model with carrying capacity 100 and that the population size is 10 when $t = 0$ and 20 when $t = 1$. Find the intrinsic growth rate.

   Use the solution to the logistic equation

$$x(t) = \frac{Kx_0}{x_0 + (K - x_0)e^{-rt}}.$$

Solutions to Exercises. Question 1 was on assignment week 10 (2004). Question 3 was on assignment week 10 (2004). Question 4 was on assignment week 10 (2004).