

MATH111 – Spring 2007

Tutorial Sheet – Week 8

This tutorial sheet covers chapters 10 of the notes.

Chapter 10. Applications of First-Order Differential Equations

Exercises

Pollution modelling in the river Thames

This exercise is based upon a workshop run by Professor G.C. Wake¹ on “Mathematics in the Environment: Modelling Pollution” at the International Conference on Mathematical Modelling and Computation (5-8th June 2006, University Brunei Darussalam). The workshop was based upon a series of mathematical models developed to analyse pollution in the River Thames (London, England) in the 1950s and 1960s. This is the simplest such model.

1. **First ‘Pond’ Model.** We consider a one-off dump of pollutant (P) into a ‘pond’, or sewage plant. A side effect of dumping pollutant is that it removes oxygen from the pond. This scenario is represented by the model

$$\frac{dP}{dt} = -k_p P, \quad P(t=0) = P_0, \quad (1)$$

$$\frac{dO_2}{dt} = -k_o P, \quad O_2(t=0) = 100\%. \quad (2)$$

In the absence of pollutant the oxygen in the pond is in equilibrium with atmospheric oxygen. The initial of 100% represents the equilibrium value, in which the oxygen has been scaled so that it has no units.

- (a) What does it mean in terms of the pond if there is a value of t such that $O_2(t) = 0$?
- (b) Solve the model. **Hint.** First solve equation (1) to find $P(t)$. Then substituted this expression into equation (2).
- (c) Sketch a graph showing how the concentration of oxygen in the pond varies as a function of time.
- (d) It is known that there is a threshold oxygen concentration in the pond, such that if the level of oxygen in the pond decreases below 30% that the fish in the pond will die. We must develop a decision-support procedure for the model.
 - (i) Show that there is a critical value of P_0 , P_{cr} such that if $P_0 \leq P_{cr}$ then $\lim_{t \rightarrow \infty} O_2(t) \geq 30\%$.
 - (ii) When $P_0 > P_{cr}$ find the time, t_{cr} , it takes for the concentration of oxygen in the pond to reach 30%.
- (e) What are suitable units for the following quantities
 - (i) pollutant concentration (P)?
 - (ii) time (t)?
 - (iii) the level of oxygen in the pond?
 - (iv) the rate constant k_p ?
 - (v) the rate constant k_o ?

In producing the final model other factors incorporated into the model included:

- The position of the sewage plant along the River Thames.

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- The effect of tidal flow.
- Different dumping strategies.

Drug absorption through a skin patch

The model discussed in this section is analysed in the paper.

Kubota, K., and Maibach, H.I. (1992). A Compartment Model for Percutaneous Absorption: Compatibility of Lag Time and Steady-State Flux with Diffusion Model. *Journal of Pharmaceutical Sciences* **81**(9), 863–865.

A drug delivery vehicle, such as a skin patch, is attached to a patient. The amount of drug per unit area (X) in the skin is described by the following differential equation

$$\frac{dX}{dt} = -(k_{12} + k_{1B})X + k_{21}A_0, \quad X(0) = 0. \quad (3)$$

In this equation A_0 is the concentration of drug per unit area in the delivery device, which is maintained at a constant level. The parameters k_{12} and k_{21} are the rate at which drug enters the vehicle from the skin and the skin from the vehicle respectively. The parameter k_{1B} is the rate at which drug leaves the skin and enters the blood stream.

1. Solve equation (3) to find the amount of drug per unit area in the skin as a function of time.
2. The cumulative amount of drug per unit area excreted from the skin (Ae_s) is expressed by

$$Ae_s = \int k_{1B}X dt, \quad Ae_s(0) = 0.$$

Determine Ae_s .

A population model

The Gompertz equation is derived from the differential equation pair

$$\begin{aligned} \frac{dN}{dt} &= \gamma N, & N(0) &= N_0 \\ \frac{d\gamma}{dt} &= -\alpha \gamma, & \gamma(0) &= \gamma_0. \end{aligned} \quad (4)$$

This system describes a population growing at an initial exponential rate of γ_0 . The growth rate itself is time dependent and decays exponentially with rate α . As the derivative approaches zero in the first equation, the theoretical population converges to a limiting value. The theoretical limit is termed the carrying capacity.

- (a) Solve system (4) to find $N(t)$.

Hint. First solve the second equation to find $\gamma(t)$. Then substitute this into the first equation. Solve the resulting differential equation to find $N(t)$.

- (b) Find the limiting population size: $\lim_{t \rightarrow \infty} N(t)$.

This question is inspired by the article.

S. Michelson, A.S. Glicksman, and J.T. Leith. Growth in solid heterogeneous human colon adenocarcinomas: comparison of simple logistical models. *Cell and Tissue Kinetics*, **20**:343–355, 1987.