MATH111 – Spring 2007
Tutorial Sheet – Week 5

This tutorial sheet covers chapter 5 and appendix D of the notes. Note that the first part of this tutorial sheet is to be detached from the tutorial sheet and handed in at the end of the tutorial.

Part One: Writing an exam question

Name ______________________________

1. Based upon the ideas and examples that are discussed in chapters 1–5 and that have been covered in the Maple worksheets write your own mid-session test question.
2. Answer your question.
3. How many marks should your question be worth? Provide a marking scheme.

Part Two: Chapter 5

Revision of key ideas

This question is to designed to help you get used to the idea of translating a word problem into a corresponding mathematical problem. You should try to solve it without looking at your lecture notes. You should discuss this question with your neighbours if you are stuck

1. Suppose that in a battle between two opposing forces each unit of army X is able to destroy b units of army Y during one time unit. Similarly each unit of army Y is able to destroy a units of army X.
   (a) Write down two **word** equations that define the problem — one for each army.
   (b) Write down, formally, the two difference equations that describe the above scenario — one difference equation for each army. Define **all** variables and explain your terms.

A worked solution to this question appears on the solutions for Assignment Week 4 (2006).

Book work questions

1. What does it mean by a fixed point x^* to be called *trivial*?
2. Consider the difference equation
$$x_{n+1} = f(x_n).$$
   Write down the conditions for a fixed point x^* of this equation to be stable and unstable, carefully defining all the terms that appear in your solution.
3. What is the importance of finding the stability of a fixed-point?
4. The fixed points of the logistic difference equation
$$x_{n+1} = rx_n(1 - x_n)$$
   are $x^* = 0$ and $x^* = (r-1)/r$. Determine their stability as a function of the parameter r. Explain the biological implications of your answer.
5. Starting from the fact that x^* is a fixed point of the difference equation
$$x_{n+1} = f(x_n)$$
   derive the linearised solution
$$\xi_n = \lambda^n \xi_0,$$
   for the growth of an initial condition given by $x_0 = x^* + \xi_0$. Describe the interpretation of the linearised solution in terms of the stability of the fixed point.
6. What information does a *response curve* contain?

Exercises

Worked solutions for the following questions appears on the solutions sheet for Assignment Week 6 (2004).

1. Consider the difference equation
$$x_{n+1} = r x_n^2 (1 - x_n).$$
   (a) Show that the fixed points of this equation are $x^* = 0$ and $x_\pm^* = \frac{r \pm \sqrt{r(r-4)}}{2r}$.
   (b) Determine the values for $x_\pm$ when $r = 5$ (correct to 5 decimal places). Calculate the corresponding eigenvalues and hence determine the stability of the two fixed points $x_\pm$.

2. The following discrete time population model has been used in the ecological literature.
$$x_{n+1} = \frac{r x_n}{1 + x_n^b}, \quad r > 0, \quad b > 1.$$
   (a) Show that this equation has at most two real fixed points: $x_1^* = 0$ and the solution of the equation $x_2^{*b} = r - 1$.
   (b) Let $f = \frac{rx}{1+x^b}$. Show that
$$\frac{\mathrm{d}f}{\mathrm{d}x} = \frac{r\left(1 + x^b [1 - b]\right)}{(1 + x^b)^2}$$
   (c) Determine the stability of the fixed point x_1^* as a function of r and b.
   (d) Show that the eigenvalue of the fixed point x_2^* is given by
$$f'(x_2^*) = (1 - b) + \frac{b}{r}$$
   Hence determine the values of r, as a function of b, for which the fixed point x_2^* is stable.

Part Three: Appendix D

1. Taylor's theorem allows you to approximate a function $f(x)$ near the value $x = a$ by
$$f(x) \approx f(a) + (x - a) f'(a) + \frac{1}{2!}(x - a)^2 f''(a) + \frac{1}{3!}(x - a)^3 f'''(a) \ldots$$
   Find the first two non-zero terms in the Taylor Series for the following functions around the point $a = 0$.
   (a) $\cos x$. $\left(1 - \frac{x^2}{2}\right)$
   (b) $\frac{\sin x}{x}$. $\left(1 - \frac{x^2}{6}\right)$
   (c) xe^x. $\left(x + x^2\right)$

2. (a) Let $y = f(x)$ and $x = g(z)$. Evaluate $\frac{\mathrm{d}y}{\mathrm{d}z}$.
   (b) Taylor's theorem allows you to approximate a function $f(x)$ near the value $x = a$ by
$$f(x) \approx f(a) + (x - a) f'(a) + \frac{1}{2!}(x - a)^2 f''(a) + \frac{1}{3!}(x - a)^3 f'''(a) \ldots \quad (1)$$
   In the derivation of the eigenvalue equation we used Taylor's Theorem to obtain the approximation
$$f(x^* + \xi) \approx f(x^*) + \xi f'(x^*) \quad (2)$$
   Explain why equation (2) follows from equation (1).