MATH111 – Spring 2007
Tutorial Sheet – Week 4

This tutorial sheet covers chapter 4 of the notes.

Chapter 4

Revision of Key Ideas

1. The average birth rate of a species per individual per year is r. The fractional death rate of the species is d per year. The initial number of animals in the population is given by N_0.
   (a) Write down a word-equation for this model.
   (b) Write down the associated discrete equation model for the size of the species population in year n.
   (c) Write down the closed-form solution to your model. How does the long-term population size depend upon the parameters r and d?
   (d) What is the flaw in the linear population model and how is it rectified?
   (e) Name two additional processes that might be included in a more general model.
2. The size of a population is given by the solution of the equation
$$y_{n+1} = f(y_n).$$
   (a) Sketch the curve $y = f(y_n)$ for a biological realistic function $f(y_n)$,
   (b) Identify the four important features that make $f(y_n)$ a realistic function to describe population dynamics.
3. Consider the difference equation
$$x_{n+1} = f(x_n), \quad x_0 = X$$
   (a) How are the fixed points of this equation found?
   (b) Why are fixed points important?
4. The population of a species is governed by the difference equation
$$x_{n+1} = f(x_n), \quad x_0 = X$$
   Explain how cobwebbing is used to determine the dynamics of this model for the specified initial condition.
5. Consider the logistic difference equation
$$x_n = r x_n (1 - x_n), \quad 0 < r < 4, \quad 0 < x_0 < 1.$$
   (a) By drawing a cobwebbing diagram show that if $0 < r < 1$ then $\lim_{n\to\infty} x_n = 0$.
   (b) What is the biological meaning of this result?

Exercises

1. Consider the following map
$$x_{n+1} = \frac{27 r x_n^2 (1 - x_n)}{16}$$
   (a) Show that if $0 \leq r \leq 4$ and $0 \leq x_n \leq 1$ then $0 \leq x_{n+1} \leq 1$.
   (b) Show that there is only one fixed point ($x^* = 0$) for $0 \leq r < \frac{64}{27}$, two fixed points when $r = \frac{64}{27}$ and three fixed points for $\frac{64}{27} \leq r \leq 4$. Give a formulae for the new pair $x_{\pm}^*$.

2. Consider the map
$$x_{n+1} = r x_n \left(1 - x_n^2\right).$$
   (a) Show that if $0 \leq x_n \leq 1$ then $0 \leq x_{n+1} \leq 1$ provided that $0 \leq r \leq \frac{3\sqrt{3}}{2}$.
   (b) Solve the fixed point equation, and show that there is only one fixed point ($x^* = 0$) for $0 \leq r \leq 1$ and three fixed points when $1 < r$. Give a formulae for the new pair $x_{\pm}^*$.
   (c) Only one of the new pair of solutions is biologically meaningful: which one is it?

Worked solutions to the exercise questions are available as follows:

1. This question appears in the questions section of Chapter 4. The worked solution appears in the corresponding appendix.
2. Assignment Week 6 (Spring 2004).