MATH 111 — Applied Mathematical Modelling I

Spring Session 2004

Mid-Session Test

Student Name: ______________________ *Student Number*: __________

Instructions

Time Allowed: 90 minutes
Number of questions: 9.

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1. Each question is to be attempted.
2. The questions are *not* of equal value. The value of each question is indicated in square brackets.
3. The examination paper is printed on both sides.
4. WORKING (including all necessary *reasoning*) is to be shown for all solutions.
5. Working is to be done in the exam paper.

Examination Materials/Aids Allowed

Non-alphanumeric calculators are permitted.

A one-page, A4-sized, double-sided summary sheet is permitted.

Examination Materials/Aids to be supplied

None.

This examination paper must NOT be removed from the examination room.

1. Consider the difference equation

$$y_{n+1} = f\left(y_n\right), \quad y_0 = 1, \quad n = 0, 1, \ldots$$

where $f\left(y_n\right)$ is some unspecified function of y_n. With reference to this equation explain what is meant by the word 'dynamics'. [1]

2. (a) Give an example of an *autonomous* difference equation and a *non-autonomous* difference equation, explaining why your equation is autonomous/non-autonomous. [2]

   (b) Identify if the following difference equations are linear or non-linear. You *must* justify your answer.[2]

   (i) $n_{y+2} = n_{y+1}y$

   (ii) $y_{n+1} = 2y_{n+1} + \sin\left(n\right)$

3. Consider the problem of modelling the number of of chickens in Mr & Mrs Tweedy's farm. Each week the following activities occur:

- The number of chickens increases through natural growth by 10%.
- A fraction, α, of the chickens are killed by foxes.
- A constant number of chickens are converted into chicken pies.

(a) Write down a **word** equation that defines this problem. [2]

(b) Write down, formally, the difference equation that describes the above scenario. Define **all** variables and explain your terms. [2]

4. How long will it take \$1500 to accumulate to at least \$2000 at 5.0% simple interest? [1]

5. Lien borrows \$20,000 to have a MATH111 chip implanted in her head so that everything makes sense. Interest is compounded monthly at 9% p.a.

   (a) In the first year Lien makes no repayments. How much does she owe at the end of the year? [1]

   (b) Starting in the second year Lien makes a repayment at the end of each month. If the loan is to be repaid after a further nine years what is the monthly repayment? [2]

6. You have been given \$1000 to invest for one year. You have a choice of three bank accounts.

   - 'You Beaut' bank offers you 10% p.a. compounded annually. At the end of the year you will pay \$20 in fees.
   - 'Fair Go' bank offers you 10% p.a. compounded quarterly. At the end of the year you will pay \$30 in fees.
   - 'Ocker' bank offers you 11% p.a. compounded every four months. At the end of the year you will pay \$25 in fees.

   Which bank should you put your money into (justify your answer)? [4]

7. Patient flow in a department of geriatric medicine is modelled by the difference equation,

$$x_n = N + (1 - \alpha - \beta - \gamma)\, x_{n-1}, \quad n = 1, 2, 3 \ldots$$

where x_n is the number of patients in the department in the nth month, N is the number of new patients admitted each month, α is the fraction of current patients who are discharged each month, β is fraction of current patients who, unfortunately, die each month and γ is the fraction of the current patients who are transferred to another section each month. For convenience we write

$$a = 1 - \alpha - \beta - \gamma$$

and assume that $0 < a < 1$.

(a) Find the general solution of the patient flow model, simplifying as far as possible. [2]

(b) What is the number of patients in the department in the limit $n \to \infty$? [2]

(c) A new geriatric ward is added to a hospital. When the ward is opened there are no patients ($x_0 = 0$).

(i) Suppose that the new ward has 100 beds and it is anticipated that 50 patients are admitted a month.

- Explain why if $a \leq 0.5$ the ward never overfills. [2]

- Give one reason why it is 'good' to operate with $a \leq 0.5$ and one reason why it is 'bad'. [2]

(ii) The ward operates with $a = 0.51$. During which month does the ward have to start turning patients away? [2]

(d) We have assumed that the parameters N & α are constant. Briefly discuss if this is reasonable. [2]

8. Consider the logistic equation with fixed harvesting

$$x_{n+1} = r x_n \left(1 - x_n\right) - h, \quad n = 0, 1, 2 \ldots$$

where $1 < r < 4$ and $0 \leq h \leq 1$.

(a) Show that harvesting is only sustainable if

$$h \leq \frac{\left(r-1\right)^2}{4r}.$$

[4]

(b) The stable fixed point of the harvesting model (should it exist) is given by

$$x^* = \frac{-(1-r) + \sqrt{(1-r)^2 - 4rh}}{2r}$$

Find the fixed point (to four decimal places), and the associated eigenvalue, when [3]

(i) $r = 1.5$ and $h = 0.015$.

(ii) $r = 1.6$ and $h = 0.05625$.

(iii) $r = 2$ and $h = 0.045$.

(c) Gollum Fresh Fish (motto 'fish fresh from the sea, three times a day') has the choice to send its fishing fleet to one of three fisheries. The value for h is regulated by Mordor Moguls. The long-term yearly profit ($\mathcal{P}$) for fishing in a fishery is

$$\mathcal{P} = ax^* - b$$

where a and b are parameters that depend upon the fishery and x^* is the steady fixed point of the harvesting model for the specified values of h and r. The numbers associated with each fishery are

Fishery one: $r = 1.5$, $h = 0.015$, $a = 3$, $b = 0.6$.

Fishery two: $r = 1.6$, $h = 0.05625$, $a = 4$, $b = 0.45$.

Fishery three: $r = 2$, $h = 0.045$, $a = 1$, $b = 0.2$.

Which fishery should Gollum Fresh Fish use and why? [2]

9. Find the fixed points of the Ricker difference equation. [2]

$$x_{n+1} = x_n \exp\left[r\left(1 - x_n\right)\right], \quad n = 0, 1, \ldots$$