MATH 111 — Applied Mathematical Modelling I

Spring Session 2003

Mid-Session Test

Student Name: ______________________ *Student Number*: __________

Instructions

Time Allowed: 90 minutes
Number of questions: 11.

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1. Each question is to be attempted.
2. The questions are *not* of equal value. The value of each question is indicated in square brackets.
3. The examination paper is printed on both sides.
4. WORKING (including all necessary *reasoning*) is to be shown for all solutions.
5. Working is to be done in the exam paper.

Examination Materials/Aids Allowed

Non-alphanumeric calculators are permitted.

A one-page, A4-sized, double-sided summary sheet is permitted.

Examination Materials/Aids to be supplied

None.

This examination paper must NOT be removed from the examination room.

1. Consider the difference equation

$$y_{n+1} = f(y_n), \quad y_0 = 1, \quad n = 0, 1, \ldots$$

where $f(y_n)$ is some unspecified function of y_n. Explain what is meant by the word 'iteration'. [1]

2.

(a) Define *autonomous* difference equation and *non-autonomous* difference equation. [2]

(b) Identify if the following difference equations are autonomous or non-autonomous. You *must* justify your answer. [3]

(i) $y_{n+2} = y_{n+1} y_n$

(ii) $y_{n+1} = 2y_{n+1} + \sin(n)$

(iii) $n_{k+2} = n_{k+1} - n_k$

3. Consider the difference equation

$$y_k = k y_{k-1}, \quad k = 1, 2, 3 \ldots$$

with initial condition $y_0 = 1$.

(a) Calculate y_1, y_2, y_3, y_4 and make a guess at the "closed-form" solution of y_k. [2]

(b) Verify that your formula satisfies the difference equation and the initial condition. [2]

4. What is the simple interest earnt on $5 000 invested for 48 months at 4.5% p.a? [1]

5. Consider the problem of modelling patient flow in a department of geriatric medicine. Each day the following activities occur:

- A number of new patients are admitted to the department for acute care.
- A fraction, α, of the current patients are treated and discharged.
- A fraction, β, of the current patients, unfortunately, die.
- A fraction of the current patients, γ, are transferred to another section.

(a) Write down a **word** equation that defines this problem. [2]

(b) Write down, formally, the difference equation that describes the above scenario. Define **all** variables and explain your terms. [2]

6. Steven decides to purchase a car for \$40 000. He has savings of \$17 000 and has the choice of two payment schemes.

   - He can put down a deposit of \$17 000 and take out a five-year loan (amortization scheme) from the bank with interest at 7.5% p.a. compounded quarterly.
   - He can put down a deposit of \$15 000 and make weekly payments of \$105 for five years to the dealer. At the end of five years he makes a final payment of \$3500

   (a) Which option should Steven choose (justify your answer)? How much money does he save? [5]

(b) Steven opts to pay the dealer directly rather than take a loan out from the bank. He decides to invest the remaining \$2 000 of his savings in a five-year term deposit account with his bank. If interest is compounded annually what is the minimum interest rate that is required for his decision to make sense? [2]

7. Find the solution of the following difference equation, simplifying as far as possible. Carefully explain each step of your solution. [4]

$$x_n - x_{n-1} = n^2, \quad x_1 = 2, \quad n = 0, 1, \ldots$$

8. Consider the population model

$$x_{n+1} = f(x_n), \quad n = 0, 1, \ldots$$

(a) What does it mean for the point x^* to be a fixed point of this equation? [1]

(b) Why is a fixed-point called a fixed-point? [1]

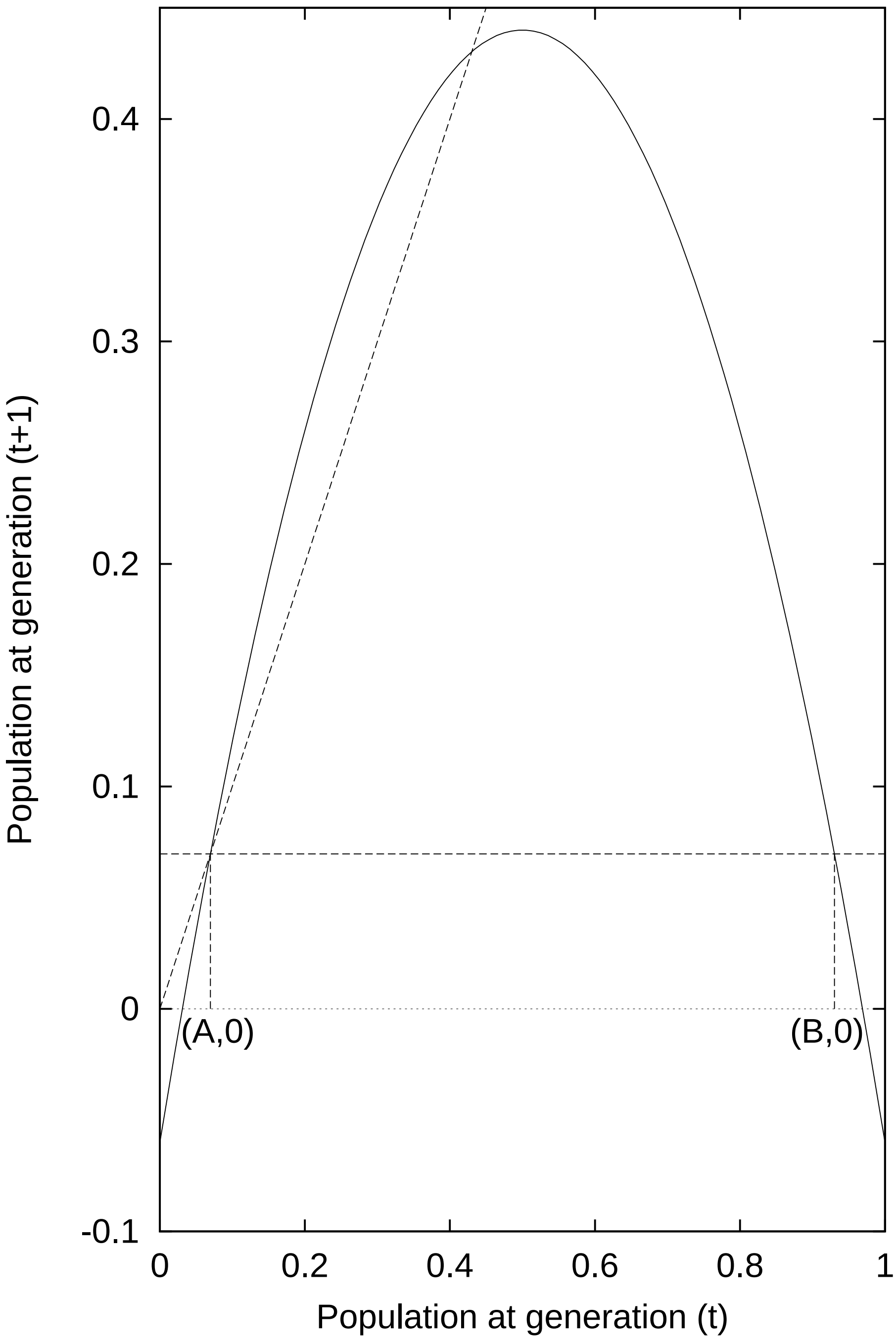
(c) Explain why it is important to find the fixed points of this equation if we want to study the long-term dynamics of this model. [1]

9. In this question we consider the discrete logistic equation with fixed harvesting

$$x_{n+1} = r x_n (1 - x_n) - h, \quad n = 0, 1, \ldots$$

with $r = 2$ and $h = 0.06$.

(a) Identify the location of the fixed point(s) of this map on the diagram below. [1]

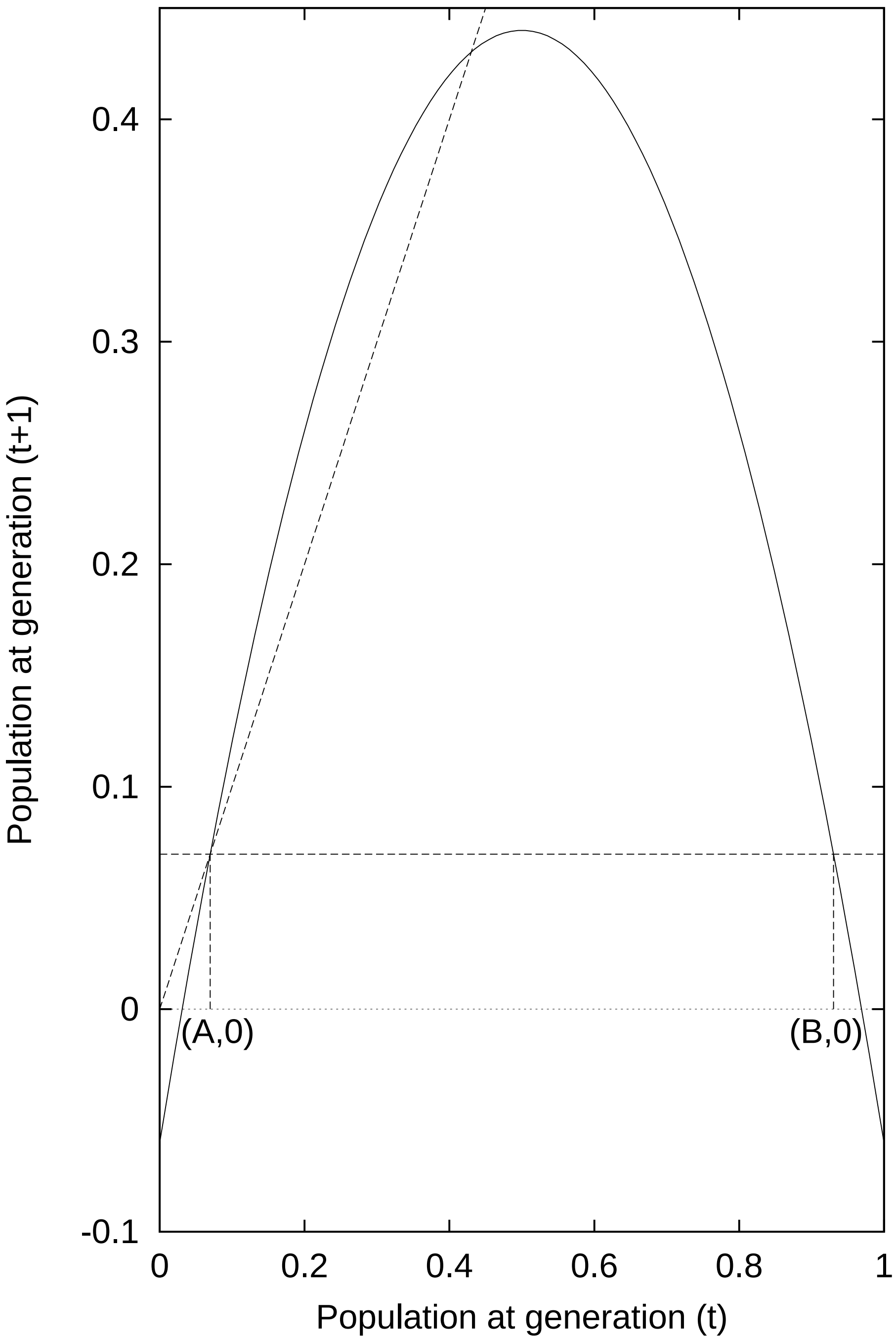


(b) By drawing successive iterations on the cobweb diagram above determine the long-term evolution of the point $x_0 = 0.8$. [1]

(c) Explain what your cobweb means biologically. [1]

(d) How would your answer to (c) change if you were choose a different value for x_0 with $x_0 \in (A, B)$? ($A = 0.0697$ and $B = 0.9303$, marked on the figure). [1]

(e) By drawing successive iterations on the cobweb diagram below determine the long-term evolution of the point $x_0 = C$, where $C \in (0, A)$, where $A = 0.0697$ is marked on the figure. [1]



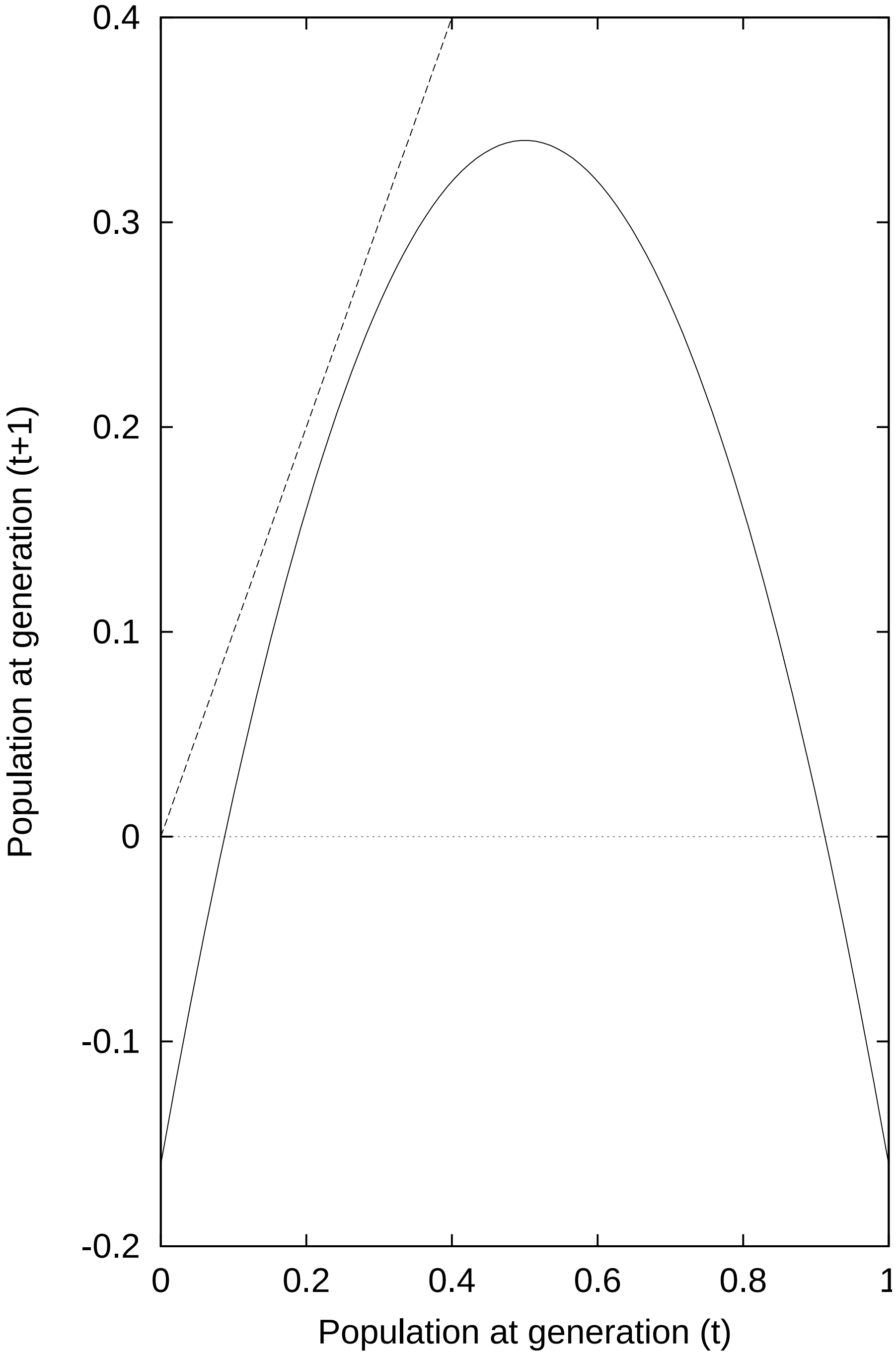
(f) Explain what your cobweb means biologically. [1]

10. In this question we consider the discrete logistic equation with fixed harvesting

$$x_{n+1} = r x_n (1 - x_n) - h, \quad n = 0, 1, \ldots$$

with $r = 2$ and $h = 0.16$.

(a) By drawing successive iterations on the cobweb diagram below determine the long-term evolution of the point $x_0 = 0.8$, [1]



(b) Explain what your cobweb means biologically. [1]

(c) How would your answer to (b) change if you were choose a different value for x_0 with $x_0 > 0$. [1]

11. Find the fixed points of the logistic difference equation. [2]

$$x_{n+1} = rx_n\left(1 - x_n\right), \quad n = 0, 1, \ldots$$

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A Formulae

A.1 First order difference equations

1. The first-order difference equation
$$x_n - ax_{n-1} = b\,(n)$$
has solution
$$x_n = x_0 a^n + \sum_{p=1}^{n} a^{n-p} b\,(p)$$

2. (a) $\displaystyle\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$

   (b) $\displaystyle\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$

   (c) $\displaystyle\sum_{k=1}^{n} k^3 = \frac{n^2(n+1)^2}{4}$

   (d) $\displaystyle\sum_{k=1}^{n} k^4 = \frac{n(n+1)(6n^3+9n^2+n-1)}{30}$

3. the sum of a geometric progression
$$k, rk, r^2k, r^3k\ldots$$
is
$$S_n = \frac{k(r^n-1)}{r-1} \quad \text{where } r \neq 1$$

A.2 Financial mathematics

In the following formulae p is the interest rate and α is the fraction of the year occupied by a conversion period.

The simple interest formula is
$$S_n = \left(1 + \frac{np}{100}\right) S_0$$
where S_n is the amount of money in the bank after n years S_0 is the amount invested.

The compound interest formula is
$$S_n = \left(1 + \frac{\alpha p}{100}\right)^n S_0$$
where S_n is the amount of money in the bank after n payments and S_0 is the amount invested.

The loan repayment (amortization scheme) formula is
$$D_n = \left(1 + \frac{\alpha p}{100}\right)^n \left(D_0 - \frac{100R}{\alpha p}\right) + \frac{100R}{\alpha p}$$
where D_n is the debt after n payments, n is the number of conversion periods, D_0 is the amount borrowed and R is the repayment made at the each of each conversion period.

The annuity formula is
$$y_n = \left(1 + \frac{\alpha p}{100}\right)^n \left(y_0 + \frac{100R}{\alpha p}\right) - \frac{100R}{\alpha p}$$
where y_n is the the amount of money in the annuity after n payments, n is the number of conversion periods, y_0 is the initial amount invested in the annuity and R is the payment made at the each of each conversion period.