School of Mathematics & Applied Statistics

MATH111: Applied Mathematical Modelling

Assignment Week 5 Spring 2007

Student Name: ________________________ *Student Number:* ____________

FULL WORKING is to be shown for all solutions.
Untidy or badly set out work will not be marked.
This assignment is to be handed in during the Wednesday lecture of week 7.

Assignment Guidelines

You are expected to structure your assignments as if writing a report for presentation to people unfamiliar with the work covered by the assignment.

Your report should be structured as follows:

1. introductory remarks
   - state purpose of assignment; and
   - state proposed tasks.
2. discussion of theory (if appropriate)
3. discussion of results
   - present outputs of each task;
   - analyse outputs of each task, including
     - comment on what your mathematical results mean in terms of the underlying physical problem.
     - comment on unexpected results;
4. concluding remarks: state whether purpose of assignment was achieved. You should answer the following questions regarding this assignment.
   - What was the most important thing that you learnt? Why was it 'important'?
   - What was the most puzzling thing that you did? (If nothing was puzzling, say so!)
5. Bibliography (if required)
6. Appendices
   - MAPLE program(s), containing comment lines explaining the purpose of your code.
   - MAPLE outputs where you think they are required to further amplify comments you have made in your report. Do *not* include every output you generated.
7. If you are not certain what is required in your report you should speak to the lecturer before you hand it in. If you don't ask, don't whinge if you lose marks because you didn't do what you should have done.

Continued on next page

School of Mathematics & Applied Statistics **MATH111**: *Applied Mathematical Modelling*
Assignment Week 5 Spring 2007 Submission Receipt

Student Name: ________________________ *Student Number:* ____________

Tutorial Class: ________ *Date Submitted:* ____________ *Tutor Initials:* ____________

Graphs and tables should be included at appropriate locations in the body of the report or as appendices at the end of the report. Please ensure all handwritten work is tidy and legible and that every page is present and in the intended order.

The grade a student receives will be the lab demonstrator's subjective assessment of how much effort that student seems to have put into creating their report. Please note that missing or incomplete outputs, inadequate discussions, and/or poor presentation will result in a low grade even if you have successfully completed all the assigned tasks. Note that marks to questions/tasks (if provided) is only indicative.

Here are some good ways to *lose* marks (5% for each one):

- No title.
- No introduction.
- No theory.
- No sections/section headings.
- Not including the model equations.
- Not discussing the model equations.
- No conclusions or summary.
- No figures.
- Inadequate referencing of sources.
- No appendices (if required).
- Repeating the questions in your report and answering them. You're supposed to write a report!
- Using the question/task numbers in your report. These don't make sense to a reader who hasn't read the assignment sheet.
- Including every graph you generated during your investigation. Summarise your findings where appropriate!
- Poor graphs: no title, no labels, too small, too large, not numbering figures etc.
- Stating that your graph uses colour, such as 'blue' and 'black' lines, but only providing a black and white graphic.
- Poor quality output: difficult to read; pages out of order.
- Not showing signs of having carried out further reading when you have been asked to read specific article(s).

This list is *not* exhaustive.

Instructions

You should work you way through this assignment, answering questions and making notes where appropriate. Where appropriate you should adapt Maple programs that you have used in previous lab sessions.

You will find it very *useful* to save any programs that you write onto a disk which you bring to subsequent labs.

1. Use a text editor such as NotePad (Programs/Accessories/NotePad) to write your program.
2. Save your program onto a disk (or alternatively onto the C drive) as a *text* file.
3. To load your program into Maple enter `read "A:/file";` where `file` is the name of your program.
4. If your program generates an error message:
   (a) Enter the command `restart;` into Maple.
   (b) Look at your code for syntax errors. Correct the code and reload it.
   (c) If you can't find your error, ask for assistance.

1 Background

The Ricker model

$$x_{n+1} = x_n \exp\left[4\left(1 - \frac{x_n}{K}\right)\right], \quad n = 0, 1, 2, \ldots \tag{1}$$

was introduced in the week 3 maple assignment. In this equation x_n is the size of the population after n breeding seasons, r is the intrinsic growth rate and K is the carrying capacity of the environment.

By defining

$$x^*_{n+1} = \frac{x_{n+1}}{K},$$

you obtained the scaled Ricker model

$$x^*_{n+1} = x^*_n \exp\left[r\left(1 - x^*_n\right)\right], \quad n = 0, 1, 2, \ldots \tag{2}$$

The long-time dynamics of the population model (2) were investigated. A variety of behaviour was found depending upon the value of the intrinsic growth rate r: period-1 one solutions were found for $r = 0.8, \&1.9$; a period-2 solution was found when $= 2.1$; a period-4 solution was found when $r = 2.6$ and chaotic behaviour was found when $r = 2.9$. In this assignment we investigate in more detail how the long-time behaviour of equation (2) depends upon the value of the per-capita reproduction rate.

The maple code used in this question is provided in appendix A. You do *not* need to retype this code. You can download it from

`http://www.uow.edu.au/~mnelson/teaching.dir/math111.dir/code.dir/bifurcation-ricker.html`

Once you have saved this code you will need to remove any non-maple text that you have downloaded. When your code runs correctly it will produce figure 1.

Figure 1 is a steady-state diagram, or bifurcation diagram, for equation (2). The code produces this figure by the following procedure.

1. It takes `n` regularly spaced values of r between the values `rstart` and `rend`.
2. For each value of r it iterates the equation `n` times.
3. All the iterates are discarded except for the last `nrep` values.
4. The `nrep` values are plotted against the value of r that generated them.

Note that if the solution is a period-1 solution and `n` is sufficiently large then there will only be one value in the array containing the last `nrep` points. Similarly if the solution is a period-2 solution (period-4 solution) then there will only be two (four) values in the array containing the last `nrep` points.

Question 1 *What's the largest periodic solution that can be identified if the array contains* `nrep` *points. Justify your answer.* [5]

2 Theory

Look closely at figure 1. As the value of r increases the graph becomes more complicated. However, at approximately $r = 3.10$ the graph suddenly becomes less clustered. A close examination of the figure will show that in this region there are only 3 values plotted on the y-axis. Thus there is a period-3 solution. The sudden de-cluttering of the steady-state diagram in this way is known as the 'period-3 window'.

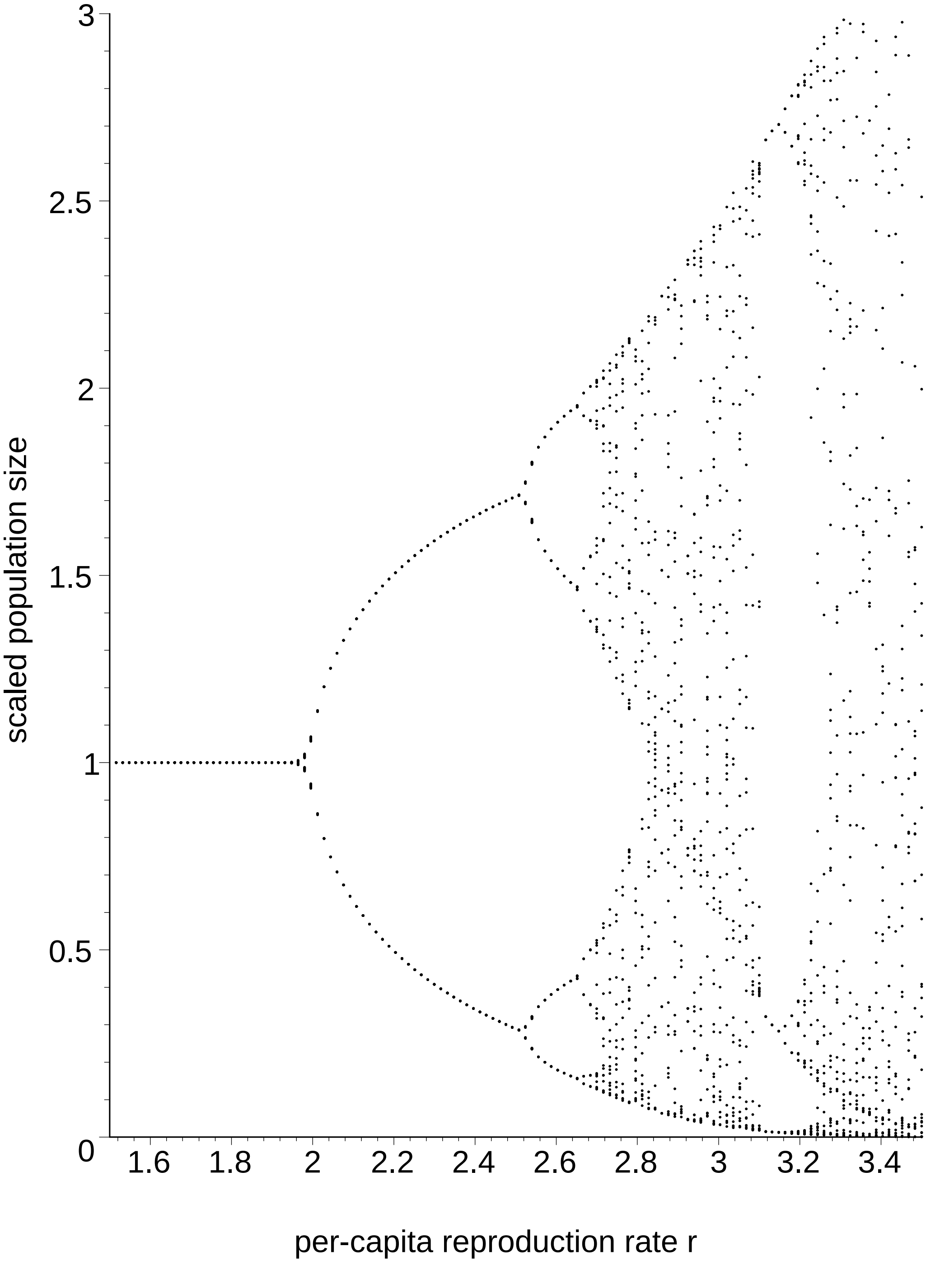

Figure 1: Computer generated steady-state diagram for equation (2) with $x_0 = 0.1$.

3 Tasks

In the following you will have re-fine the bifurcation diagram produced by the maple code. This may involve: changing the range of values of r over which the diagram is plotted, by changing the values of `rstart` and `rend`; increasing the number of points that are iterated (`n`); changing the number of representative points that are plotted (`nrep`); and changing the range of y values over which your graphs are plotted, by changing the values of `ystart` and `yend`.

For any question in which you use a bifurcation diagram "estimate the values of r over which" you should include your diagram as part of your answer. The caption for this figure should state the values of the parameters `n` and `nrep` that you used to generate it.

For some of the following questions you will be asked to use the code in appendix B. In answering these questions you should note that we are only interested in the 'long-time' behaviour of the population and adapt the code accordingly.

Question 2 *In which paper is the Ricker model first suggested? (Hint, read the article by May (1976).)*

What was the title of the paper? (Hint. When you have answered the first part of this question you may need to speak to a librarian). [5]

1. Find the fixed points of the Ricker model and determine their stability as a function of r. Plot this information in a steady-state diagram. (You may draw the steady-state diagram by hand). [5]

2. (a) Generate a high-quality bifurcation diagram with $1 \leq r \leq 4$. [3]
   Hint. You may need to change the range on the y-axis.

   (b) By refining your bifurcation diagram estimate the values of r over which a period-2 is found. Estimate the two values of r to two decimal places. [2]

   (c) Using a value of r in this region show that there is a period-2 solution by using the code in appendix B. [2]

   Hint. You will need to change the definition of the function `f` in this code.

3. (a) By refining your bifurcation diagram estimate the values of r over which a period-4 solution is found. Estimate the lower and upper values of r to two and three decimal places respectively. [2]

   (b) Using a value of r in this region show that there is a period-4 solution by using the code in appendix B. [2]

4. (a) By refining your bifurcation diagram find a value of r over which you would expect a period-8 solution to exist. Estimate the two values of r to three decimal places. [2]

   (b) Using this value of r show that there is a period-8 solution by using the code in appendix B. [2]

5. (a) By refining your bifurcation diagram find a value of r over which you would expect a period-16 solution to exist. Estimate the two values of r to three decimal places. [2]
   Hint. It will not longer be useful to have a figure going from `ystart :=0` to `yend := 3`. You will need to change the values of these variables to get the best possible picture showing the period-8 to period-16 transition.

   (b) Using this value of r show that there is a period-16 solution by using the code in appendix B. [2]

6. We have found the range of r over which the period-1 solution is stable, the period-2 solution is stable, the period-4 solution is stable, the period-8 solution is stable and the period-16 solution is stable. What do you notice about the range of stability of these solutions? [2]

7. Discuss your answer to the last question within the context of the Feigenbaum Constant. You will need to find a definition of the Feigenbaum Constant. You will need to cite your source in your report. [2]

   You may find it useful to change the range of y values over which your graphs are plotted.

8. (a) By refining your bifurcation diagram estimate the values of r over which there is a period-3 solution. [2]

   (b) Using a value of r in this region show that there is a period-3 solution by using the code in appendix B. [2]

9. By refining your bifurcation diagram in the region $2.76 < r < 2.79$ determine the periodicity of the window, n, that appears in this region. Using an appropriate value of r in this region show an example of a period-n solution. [4]

10. By refining your bifurcation diagram in the vicinity of the point $r = 3.60$ determine the periodicity of the window, n, that appears in this region. Using an appropriate value of r in this region show an example of a period-n solution.

    Hint, for the first part of this question it will be useful to generate one steady-state diagram with `ystart :=0` `yend :=1` and a second steady-state diagram with `ystart :=1` and `yend := 4.0`. In fact, to generate *really* good diagrams you will need to play around with these numbers. [4]

4 Further reading

The introduction and discussion that you write for this assignment should be informed by the content of the paper by May (1976). You should make it clear in your report what you have learnt from reading this article.

May, R.M. 1976. Simple mathematical models with very complicated dynamics. *Nature*, **261**: 459–467.

5 Marking

Every student starts with a mark of 100. The questions and tasks for this assignment are worth 50 marks. Every time your answer to a question or task is incomplete or wrong you lose marks. In addition to losing marks in this way can also lose marks for a badly written report. Although, in theory, there are 50 marks attached to the report there is no upper bound on the number of marks you can lose. If you make 17 bad mistakes (see the list on page two) then you will lose 85 marks. However, your mark will not be reduced below zero.

A Maple code: bifurcation.maple

You do *not* need to retype this code. You can download it from

`http://www.uow.edu.au/~mnelson/teaching.dir/math111.dir/code.dir/bifurcation.html`

B Maple code: ricker.maple

You do *not* need to retype this code. You can download it from

`http://www.uow.edu.au/~mnelson/teaching.dir/math111.dir/code.dir/ricker.html`

```
# ricker.maple (05.08.07)
# A simple maple program to iterate the scaled Ricker model
# x_{n+1} = x_{n}*exp[r*(1-x_n)];
#
# 21.08.07 Revised to add a utility useful in examining
#          periodic solutions

finalyear := 500;    # The final value of 'n'.
r         := 2.656;   # static birth rate.
year      := n->n;  # define the 'time' variable.

f := x -> x*exp(r*(1-x)); # x_{n+1} = f(x_{n})

# Instead of using the variable x_{n} we will use the variable
# pop{n}
pop  := proc(n)          # define the values of pop(n) recursively.
```

```
        option remember; # Note that using the option remember causes the
        f(pop(n-1))      # previous values pop(n-1) to be retained so that
        end:             # subsequent values may be based on them.

pop(0) := 0.1;          # the initial population size.

# calculate the ordered pairs (pop_n,time_n) for n=0..finalyear

solution := [seq([year(n),pop(n)],n=0..finalyear)]:
# if you want to see the values of the data points remove the hash
# at the start of the next line
# array(solution):

# plot ALL the data points
plot(solution,style=POINT,symbol=BOX,color=BLACK,\
     labels=["generations","population size"],\
     labeldirections=[horizontal,vertical]);
#
# Sometimes we don't want to plot ALL of our data: We might not
# want to show any 'transient' behaviour, just the long-term
# 'steady' behaviour.  The following piece of code pulls out
# the population size for the last 10 generations.
#
lookback := 10;
solution2 := [seq([solution[n,1],solution[n,2]],n=finalyear-lookback..finalyear)]:

# plot the last "lookback" points
p1 := plot(solution2,style=POINT,symbol=BOX,color=BLACK):
p2 := plot(solution2,style=LINE,color=BLACK):

# the following code plots the population in year n as a function
# of the population size in the previous year. It is useful when
# you have a periodic orbit.
#
solution3 := [seq([solution[n,2],solution[n-1,2]],n=finalyear-lookback..finalyear)]:

p3 := plot(solution3,style=POINT,symbol=BOX,color=RED):
# If you have a period n solution then there should be 'n' points on
# this figure - but you might need to increase lookback.
p4 := plot(solution3,style=LINE,color=BLUE):


with(plots):
display({p1,p2});
display({p3});
display({p3,p4});

pop := 'pop':
```

Note that this code produces four figures. The last figure will contain one point if the solution is a period-one solution, two points if the solution is a period-two solution, three points if the solution is a period-three solution etc. You will need to ensure that the value for the `lookback` variable is greater than the periodicity of the solution.