School of Mathematics & Applied Statistics

MATH111: Applied Mathematical Modelling
Assignment Week 3 Spring 2007

Student Name: ______________________________ *Student Number:* ____________

FULL WORKING is to be shown for all solutions.
Untidy or badly set out work will not be marked.
This assignment is to be handed in during the Wednesday lecture of week 5.

Assignment Guidelines

You are expected to structure your assignments as if writing a report for presentation to people unfamiliar with the work covered by the assignment.

Your report should be structured as follows:

1. introductory remarks
   - state purpose of assignment; and
   - state proposed tasks.
2. discussion of theory (if appropriate)
3. discussion of results
   - present outputs of each task;
   - analyse outputs of each task, including
     - comment on what your mathematical results mean in terms of the underlying physical problem.
     - comment on unexpected results;
4. concluding remarks: state whether purpose of assignment was achieved. You should answer the following questions regarding this assignment.
   - What was the most important thing that you learnt? Why was it 'important'?
   - What was the most puzzling thing that you did? (If nothing was puzzling, say so!)
5. Bibliography (if required)
6. Appendices
   - MAPLE program(s), containing comment lines explaining the purpose of your code.
   - MAPLE outputs where you think they are required to further amplify comments you have made in your report. Do *not* include every output you generated.
7. If you are not certain what is required in your report you should speak to the lecturer before you hand it in. If you don't ask, don't whinge if you lose marks because you didn't do what you should have done.

Continued on next page

School of Mathematics & Applied Statistics **MATH111**: *Applied Mathematical Modelling*
Assignment Week 3 Spring 2007 Submission Receipt

Student Name: ______________________________ *Student Number:* ____________

Tutorial Class: ________ *Date Submitted:* ____________ *Tutor Initials:* ____________

Graphs and tables should be included at appropriate locations in the body of the report or as appendices at the end of the report. Please ensure all handwritten work is tidy and legible and that every page is present and in the intended order.

The grade a student receives will be the lab demonstrator's subjective assessment of how much effort that student seems to have put into creating their report. Please note that missing or incomplete outputs, inadequate discussions, and/or poor presentation will result in a low grade even if you have successfully completed all the assigned tasks. Note that marks to questions/tasks (if provided) is only indicative.

Here are some good ways to *lose* marks (5% for each one):

- No title.
- No introduction.
- No theory.
- No sections/section headings.
- Not including the model equations.
- Not discussing the model equations.
- No conclusions or summary.
- No figures.
- Inadequate referencing of sources.
- No appendices (if required).
- Repeating the questions in your report and answering them. You're supposed to write a report!
- Using the question/task numbers in your report. These don't make sense to a reader who hasn't read the assignment sheet.
- Including every graph you generated during your investigation. Summarise your findings where appropriate!
- Poor graphs: no title, no labels, too small, too large, not numbering figures etc.
- Stating that your graph uses colour, such as 'blue' and 'black' lines, but only providing a black and white graphic.
- Poor quality output: difficult to read; pages out of order.
- Not showing signs of having carried out further reading when you have been asked to read specific article(s).

This list is *not* exhaustive.

Instructions

You should work you way through this assignment, answering questions and making notes where appropriate. Where appropriate you should adapt Maple programs that you have used in previous lab sessions.

You will find it very *useful* to save any programs that you write onto a disk which you bring to subsequent labs.

1. Use a text editor such as NotePad (Programs/Accessories/NotePad) to write your program.
2. Save your program onto a disk (or alternatively onto the C drive) as a *text* file.
3. To load your program into Maple enter `read "A:/file";` where `file` is the name of your program.
4. If your program generates an error message:
   (a) Enter the command `restart;` into Maple.
   (b) Look at your code for syntax errors. Correct the code and reload it.
   (c) If you can't find your error, ask for assistance.

1 Background

Many animal populations have distinct breeding seasons. If we assume that the population only changes during the breeding session, which occurs at well defined discrete intervals, then the population dynamics can be modelled using a *difference equation*, a *discrete* model, rather than a differential equation, a *continuous* model. In this assignment we will learn that a simple difference equation model can predicate a variety of growth patterns in a population.

We assume that the only processes that are acting to change the population size are births and deaths.

Question 1 (5 marks) *Write down a* ***word*** *equation that defines this problem.* □

In this assignment we investigate a particular model for the dynamics of a population known as the Ricker model. The Ricker model is given by

$$x_{n+1} = x_n \exp\left[r\left(1 - \frac{x_n}{K}\right)\right], \quad n = 0, 1, 2, \ldots \tag{1}$$

In this equation x_n the size of the population after n breeding seasons, r is the intrinsic growth rate and K is the carrying capacity of the environment.

Question 2 (5 marks) *The scaled Ricker model. is obtained by introducing a new variable, x^*_{n+1}, defined by*

$$x^*_{n+1} = \frac{x_{n+1}}{K}.$$

1. *Derive the scaled Ricker model*

$$x^*_{n+1} = x^*_n \exp\left[r\left(1 - x^*\right)\right], \quad n = 0, 1, 2, \ldots \tag{2}$$

2. *What is the mathematical attraction in studying equation* (2) *rather than equation* (1)*?*

In this assignment we solve equation (2) iteratively using maple. We investigate how the population dynamics change as the parameters r and x^*_0 are varied. A variety of growth patterns will be found as the value of r is increased: fixed-points through periodic solutions to chaotic behaviour.

The *numerical* results that you *discover* during this assignment will motivate the mathematical techniques that we will subsequently develop.

The maple code used in this assignment is provided in appendix A. When you have typed in this code you will produce figure 1.

2 Theory

Suppose that for a large enough value of n the solution to equation (2) takes the value p in the nth year and p in the $n+1$th year, i.e. the long-term size of the population is a pattern ppp in which the size of the population is repeated every year. Such a solution is called a fixed-point or a period-one solution.

Suppose that for a large enough value of n the solution to equation (2) takes the value p in the nth year, q in the $n+1$th year ($p \neq q$, p in the $n+2$th year and q in the $n+3$th year i.e. the long-term size of the population is a pattern $pqpq$ in which the size of the population is repeated every *second* year. Such a solution is called a period-two solution.

Suppose that for a large enough value of n the solution to equation (2) is a pattern $pqrspqrs$ (with p, q, r and s distinct values) in which the size of the population is repeated every *fourth* year. Such a solution is called a period-four solution.

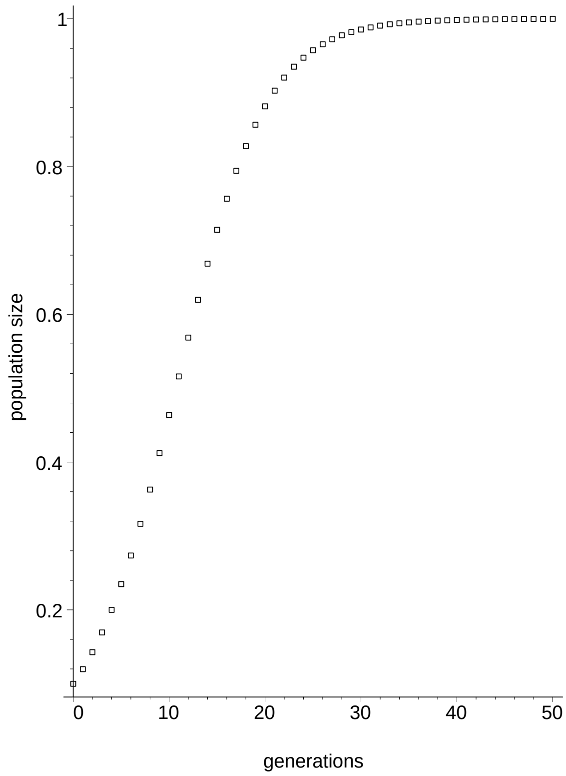

Figure 1: Computer generated solution for equation (2) with $x_0^* = 0.1$, $r = 0.2$.

3 Tasks

In investigating the solution of equation (2) we are interested in the 'long-time' behaviour of the population, i.e. the values of x_n^* when n is "sufficiently large". How large does n have to be for it to be "sufficiently large"? You will have to discover that by trial and error, it will depend upon the values used for r and x_0^*.

1. What is the order of equation (2)? Is it linear or non-linear? Is it autonomous or non-autonomous? Justify your answers. [3]

2. (a) Determine the long-term behaviour of the population when $r = 0.8$. What does this behaviour represent biologically? [2]

   (b) Describe the long-term behaviour of the population when $r = 1.9$. What does this behaviour represent biologically? This kind of solution shows 'damped oscillations'. What are 'damped oscillations'? [3]

3. (a) Determine the long-term behaviour of the population when $r = 2.1$. What does this behaviour represent biologically? [2]

   (b) The behaviour of the model when $r = 2.1$ is quite different to that when $r = 1.8$. By varying the value of r determine to two decimal places a critical value of r, r_{cr} such that if $r < r_{\text{cr}}$ the long-term behaviour of the model is biologically the same as when $r = 1.8$ whereas if $r > r_{\text{cr}}$ the long-term behaviour of the model is biologically the same as when $r = 2.1$.

   You must supply appropriate data/graphs to substantiate your critical value of r [4]

4. Describe the long-term solution to equation (2) when

   (a) $r = 2.6$.

   (b) $r = 2.9$.

   Hint You may need to significantly increase the parameters `finalyear` and `lookback`. [5]

5. We have used the initial condition $x_0 = 0.1$. Can we ever know the exact size of the initial population? No! Do our predictions of the long-term population size depend upon the choice of initial condition? That's a good question! For each of the following values of r ($r = 0.2$, $r = 0.8$, $r = 1.9$, $r = 2.1$, $r = 2.6$ and $r = 2.9$) compare the long-term behaviour of the model for the following pairs of initial conditions.

   (a) $x_0 = 0.1$ and $x_0 = 0.101$.

   (b) $x_0 = 0.1$ and $x = 0.180$.

   Discuss the possibility of making an accurate long-term prediction for the size of the population given an uncertainty in the value of the initial condition.

   Instead of showing two figures for each value of r use the `display` function to plot your data as one figure. (Look at how we did this in the maple sheet for week two. You should be able to make use of the unused variable names `p3` and `p4` in the code.)

   Note. In writing your report you should *summarise* your findings to this question, including an illustrative figure where appropriate. Do not include every figure that you generate. [19]

6. What does the word *dynamics* mean? How does the meaning of this word relate to what we have studied in this assignment? [2]

4 Further reading

The introduction and discussion that you write for this assignment should be informed by the content of the paper by May (1976). You should make it clear in your report what you have learnt from reading this article. This article is available through the library catalogue as an electronic reading for MATH111.

May, R.M. 1976. Simple mathematical models with very complicated dynamics. *Nature*, **261**: 459–467.

5 Marking

Every student starts with a mark of 100. The questions and tasks for this assignment are worth 50 marks. Every time your answer to a question or task is incomplete or wrong you lose marks. In addition to losing marks in this way can also lose marks for a badly written report. Although, in theory, there are 50 marks attached to the report there is no upper bound on the number of marks you can lose. If you make 17 bad mistakes (see the list on page two) then you will lose 85 marks. However, your mark will not be reduced below zero.

A Maple code

```
# ricker.maple (05.08.07)
# A simple maple program to iterate the scaled Ricker model
# x_{n+1} = x_{n}*exp[r*(1-x_n)];

finalyear := 50;    # The final value of ‘n’.
r         := 0.2;   # static birth rate.
year      := n->n;  # define the ‘time’ variable.

f := x -> x*exp(r*(1-x)); # x_{n+1} = f(x_{n})

# Instead of using the variable x_{n} we will use the variable
# pop{n}
pop  := proc(n)          # define the values of pop(n) recursively.
        option remember; # Note that using the option remember causes the
        f(pop(n-1))      # previous values pop(n-1) to be retained so that
```

```
        end:              # subsequent values may be based on them.

pop(0) := 0.1;          # the initial population size.

# calculate the ordered pairs (pop_n,time_n) for n=0..finalyear

solution := [seq([year(n),pop(n)],n=0..finalyear)]:
# if you want to see the values of the data points remove the hash
# at the start of the next line
# array(solution):

# plot ALL the data points
plot(solution,style=POINT,symbol=BOX,color=BLACK,\
     labels=["generations","population size"],\
     labeldirections=[horizontal,vertical]);
#
# Sometimes we don't want to plot ALL of our data: We might not
# want to show any 'transient' behaviour, just the long-term
# 'steady' behaviour.  The following piece of code pulls out
# the population size for the last 10 generations.
#
lookback := 10;
solution2 := [seq([solution[n,1],solution[n,2]],n=finalyear-lookback..finalyear)]:

# plot the last "lookback" points
p1 := plot(solution2,style=POINT,symbol=BOX,color=BLACK):
p2 := plot(solution2,style=LINE,color=BLACK):
#p3 := plot(solution2,style=POINT,symbol=CROSS,color=RED):
#p4 := plot(solution2,style=LINE,color=RED):

with(plots):
display({p1,p2});

pop := 'pop':
```