5 Linear Stability Analysis of Fixed-Points

5.1 Aims

After working through this chapter you will

1. Be able to explain the importance of the concept of 'stability' when discussing fixed points.
2. Understand the ideas behind linear stability analysis.
3. Know that a fixed point x^* of the difference equation

$$x_{n+1} = f(x_n)$$

is *stable* if

$$-1 < f'(x^*) < 1$$

and *unstable* if

$$|f'(x^*)| > 1$$

4. Know how the stability of the two fixed points to the logistic map

$$x_{n+1} = rx_n(1 - x_n)$$

depend upon the value of the parameter r.

5.2 Physical motivation of 'stability'

Consider the difference equation

$$x_{n+1} = f(x_n)\,. \qquad (5.1)$$

The fixed-point(s) (also known as period-1 solutions or equilibrium values or steady-state solutions) of eq^n (5.1) are given by the solution(s) of the eq^n

$$x^* = \_\_\_\_\_\_.$$

At a fixed-point the system does *not* change with time. (W7)

By carefully examining what happens to a system when it is *near* a fixed-point we can gain an insight into the behaviour of equation (5.1).

To do this we distinguish between two types of fixed points by introducing the the concept of *stability* ______. Since this is best described by analogy, see figure 5.1, which exemplifies three situations, two of which are steady states; one steady stable is stable, the other unstable.

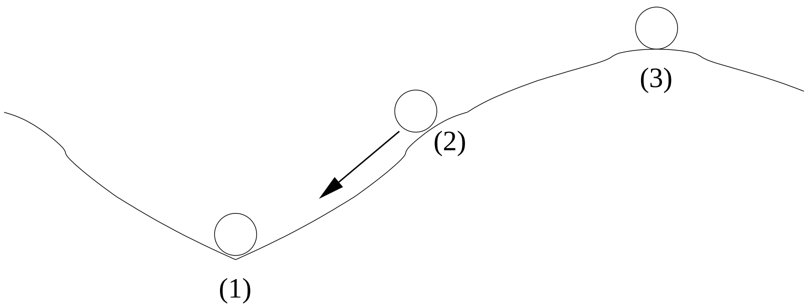

Figure 5.1: In this landscape balls one and three are at rest and represent steady-state situations. Which is *stable* and which is *unstable*? Why? Ball two is not in a steady state, since its position and speed are continuously changing. (After Edelstein-Keshet, 1988)

A steady state is termed *stable* if neighbouring states are attracted to it and *unstable* if the converse is true.

Question 5.1 *Explain how figure 5.1 illustrates the principle of stability.*

As shown in figure 5.1 balls one and three are at rest and represent steady-state situations. Ball one is stable; if moved slightly it will return to its former position. Ball three is unstable. The slightest disturbance will cause it to fall into one of the adjoining valleys — it will not return to this position if disturbed slightly.

Such distinctions have practical implications in biology. When steady states are unstable, great changes may be about to happen: a population may crash, possibly becoming extinct, or a disease may spread through a population.

This *qualitative* information about whether change is imminent is potentially of *great importance*.

With this motivation behind us, we turn to the analysis that permits us to make such predictions.

5.3 Linear stability analysis

Suppose that $x^*$is a fixed point of equation (5.1). We now proceed to explore its stability by asking the following key question: given some value x_0 *close* to x^*, will iterations of this initial point, i.e. the values x_n, tend toward or away from this steady state? To answer this question we write

$$x_0 = x^* + \xi_0,$$

where ξ_0 is the distance between the initial population size and the fixed point. If $\xi_0 > 0$ then $x_0 > x^*$ and conversely if $\xi_0 < 0$ then $x_0 < x^*$. In either case we assume that $|\xi| \ll 1$.

Using eq$^{\mathrm{n}}$ (5.1) we can calculate the size of the population at the next generation.

$$x_1 = f(x_0).$$

We can rewrite x_1 in terms of the distance between it and the fixed point

$$x_1 = x^* + \xi_1.$$

We have

$$x_{n+1} = x^* + \xi_{n+1}, \qquad (5.2)$$

$$= f(x_n) \text{(why?)}$$

$+ \xi_n$ (why?)

$\xi_n^2)$;

(why?)

(5.3)

Combining equations (5.2) and (5.3) we have

$$x^* + \xi_{n+1} = f(x^*) + \xi_n f'(x^*),$$

from which we obtain

$$\xi_{n+1} = \xi_n f'(x^*). \qquad \text{(why?)} \tag{5.4}$$

$$= \lambda \xi_n, \tag{5.5}$$

$$\text{where} \quad \lambda = \underline{f'(x^*)} \tag{5.6}$$

Thus we have obtained a <u>*linear*</u> equation for the distance between the the nth iteration of our <u>initial</u> <u>point</u> (x_n) and the <u>fixed</u> <u>point</u> x^*. In this equation the parameter λ is known as the <u>*eigenvalue*</u> of the fixed point x^*.

The solution of equation (5.5) is

$$\xi_{n+1} = \underline{\qquad}$$

$$\Rightarrow \lim_{n\to\infty} \xi_{n+1} = \begin{cases} - \\ \underline{\qquad} \end{cases} \text{if} \begin{cases} \underline{\qquad\quad} \\ \underline{\qquad\quad} \end{cases}$$

Thus the <u>eigenvalue</u> determines the <u>stability</u> of the steady-state.

Definition 5.1 ***(Stable and unstable fixed point)*** *A fixed point x^* is*

$$\textit{stable} \quad \textit{if} \qquad -1 < \quad f'(x^*) \quad < 1 \tag{5.7}$$

$$\textit{unstable} \quad \textit{when} \qquad |f'(x^*)| \quad > 1 \tag{5.8}$$

Note that if $\lambda = \pm 1$ then our 'linear stability analysis' does not determine the stability of the fixed point x^*. In these cases further analysis is required to determine the stability of the fixed point.

The stability of a fixed point x^* is determined by the value of $f'(x^*)$.

Example 5.1 *Suppose that the number of deers (in thousands) in a forest can be modeled by the difference equation*

$$p_{n+1} = 1.5p_n - 0.5p_n^2.$$

(a) Determine the fixed points of this model.

$$p^* = 1.5p^* - 0.5{p^*}^2,$$

$$\Rightarrow 0 = 0.5p^* - 0.5{p^*}^2,$$

$$0 = 0.5p^* \left(1 - p^*\right).$$

The fixed points are therefore $p_1^* = 0$ *and* $p_2^* = 1$.

Example 5.2 *(b) Determine the stability of the fixed point(s).*

The stability of a fixed point p^* is determined by its eigenvalue

$$\lambda = f'(x^*) \, .$$

For this problem we have

$$f(p) = 1.5p - 0.5p^2,$$
$$f'(p) = 1.5 - p.$$

For the fixed point $p_1^* = 0$ we have $\lambda_1 = 1.5$. As $|\lambda_1| > 1$ the fixed point is unstable.

For the fixed point $p_2^* = 1$ we have $\lambda_2 = 0.5$. As $|\lambda_2| < 1$ the fixed point is stable.

Question 5.2 *Is the distinction between a stable fixed point and an unstable fixed point of practical importance?*

Question 5.3 *Why does* $|\lambda| < 1$ *mean that the fixed point* x^* *is stable.*

5.4 Stability of the fixed points in the logistic equation

For the logistic map we have

$$x_{n+1} = x_n (1 - x_n) . \qquad (5.9)$$

Thus

$$f(x) = rx(1-x)$$

and

$$\frac{\mathrm{d}f(x)}{\mathrm{d}x} = \underline{\qquad\qquad}.$$

The fixed points of the logistic map are given by

$$f(x) = x,$$
$$rx(1-x) = x,$$

$$x = 0 \quad \text{or} \quad x = \underline{\qquad}$$

The fixed points and corresponding eigenvalues λ are (a little room for your calculation)

$$x_1^* = 0, \qquad \lambda = f'(x_1^*) = \underline{r}, \qquad (5.10)$$
$$x_2^* = \underline{\frac{r-1}{r}}, \qquad \lambda = f'(x_2^*) = \underline{2-r.} \qquad (5.11)$$

In the range $0 \leq r \leq 1$ the trivial fixed point is the only biologically meaningful, that is non-negative fixed-point.

This point is stable when $\underline{r < 1}$.

The trivial fixed point is unstable if $\underline{r > 1}$.

The second fixed point is <u>*stable*</u> in the range $\underline{1 < r < 3}$, but <u>*unstable*</u> in the range $\underline{r > 3}$.

Indeed both fixed points are <u>unstable</u> in the later range.

The stability information for period-1 solutions of the logistic difference equation is summarised in figure 5.2. A plot such as figure 5.2 which shows how the steady-state solutions and their stability varies with a parameter is often referred to as a *response curve* or a *steady-state diagram* or a *bifurcation diagram*.

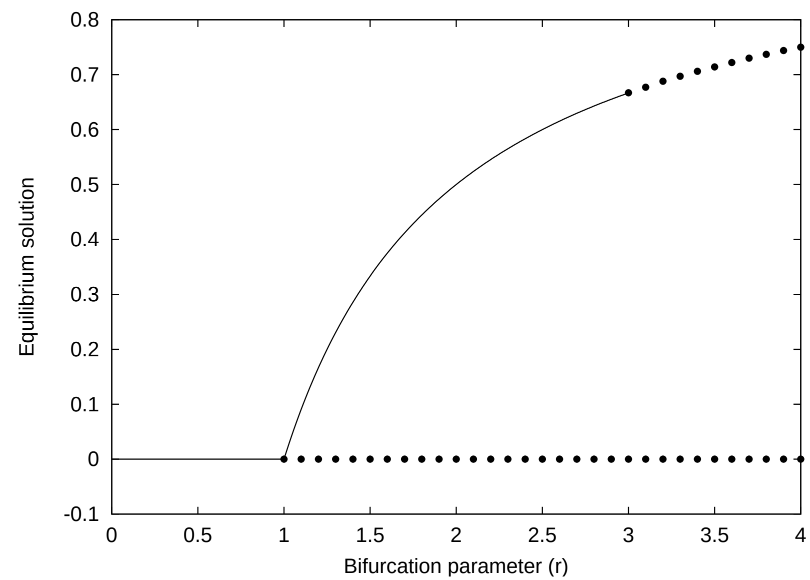

Figure 5.2: Period-1 solutions to the logistic difference eq^n and their stability: a solid line indicates a stable sol^n whilst a doted line indicates an unstable sol^n.

5.5 Fixed points, their stability and bifurcations

Definition 5.2 (Bifurcation) *Consider the equation*

$$x_{n+1} = f\left(x_n, r\right)$$

Loosely speaking a 'bifurcation' is a value of the parameter r at which 'the number of solutions' to the equation changes.

Suppose that x^* is a *stable* fixed point of the discrete model

$$x_{n+1} = f\left(x_n\right).$$

Stability can be lost in two ways: the eigenvalue ($\lambda = f'\left(x^*\right)$) can increase through 1 or decease through -1.

The former is known as a *tangent bifurcation* whilst the later is known as a *pitchfork bifurcation* (or *period-doubling bifurcation*).

Question 5.4 *Based upon your numerical work what kind of solution do you have in the logistic difference equation when r increases through the value 3?*

The study of fixed-points (i.e. period-one solutions), their stability and their bifurcations is the process through which the properties of non-linear discrete differences are elucidated in a systematic manner. However, these issues are left for another course.

5.6 Revision of key ideas

5.7 Concept map

Draw a concept map for this chapter relating the aims/key ideas of the chapter. If you are unfamiliar with the idea of a concept map see appendix A.