1 Introduction, Definitions and Basic Concepts

1.1 Aims

After working through this chapter you will be able to:

1. recognise difference equations and have an appreciation of circumstances when they arise in physical problems;
2. undertake the process of iteration to obtain solutions to difference equations;

3. understand what the following terms mean in relation to difference equations: state, order, initial conditions;
4. see the link between the order of a difference equation and the number of initial conditions;
5. distinguish between linear and non-linear difference equations *and* autonomous and non-autonomous systems;
6. verify if a given solution is the solution to the difference equation;
7. guess a solution to a difference equation by observing the patterns of the sequence of solutions;

1.2 Overview of Discrete Systems

Often a physical setting is reduced to a set of measurements made at a sequence of *equally spaced* specified times. For instance, the price of a stock might be monitored once an hour, the temperature at a weather station might be recorded once a day and the population size of an animal species might be estimated once a season.

In this part of the course we are interested in studying the **change over time** (or **dynamics**) of a quantity (such as the size of an animal population, the amount of money in a bank account, the temperature at a fixed point...etc.) which is modelled as a discrete system and is considered to occur at regular, **equally space time intervals** (such as one year, half year, one season, one month, ten days, one day, one hour...etc.).

Note: In other physical problems, it may be more appropriate to model the particular system as a continuous problem rather than a discrete case. For example:- the growth of a plant, radioactive decay, a cup of hot coffee cooling...etc.

Discrete systems are modelled by *difference equations*, whereas **continuous systems** are described by *differential equations.*

1.3 Examples of Difference Equations

1.3.1 Banking

Example 1.1 *The interest on an investment of \$5 000 at the MAS Bank earns 6.5% interest rate compounded monthly. The new principal is given by the equation*

$$P_{n+1} = \left(1 + \frac{0.065}{12}\right) P_n \tag{1.1}$$

$$P_0 = 5\,000, \quad n = 0, 1, 2, \ldots \tag{1.2}$$

Equation 1.1 is an example of a difference equation. In this equation the subscript n is a non negative integer which in this case represents the number of months after the investment is made. So $n = 1$ is a month after the investment is made, $n = 2$ is two months after the investment is made and so on.

The notation P_0 is the initial amount in the bank, and P_1 is the amount in the bank after the first month, and so on. So after the 1st month, the initial investment of \$5 000 would have grown to

$$\begin{aligned} P_1 &= \left(1 + \frac{0.065}{12}\right) P_0, \\ &= \left(1 + \frac{0.065}{12}\right) \underline{\hspace{3cm}} \\ &\approx \underline{\hspace{3cm}} \end{aligned}$$

In the second month

$$P_2 = \left(1 + \frac{0.065}{12}\right) P_1,$$
$$= \left(1 + \frac{0.065}{12}\right) \underline{\hspace{3cm}}$$
$$\approx \underline{\hspace{3cm}}$$

and so on.

This process of repetition is called <u>iteration</u>. All difference equations can be solved via <u>iteration</u> as long as appropriate starting values, known as <u>initial conditions</u>, are provided. In the example above, $P_0 = 5\,000$ is the <u>initial condition</u> to the difference equation.

Question 1.1 *Equation (1.1) can be written as*

$$P_{n+1} = P_n + \frac{0.065}{12} P_n \qquad (1.3)$$

In this equation what do the terms P_n *and* $\frac{0.065}{12} P_n$ *represent?*

1.3.2 Population biology (Fibonacci Numbers)

Example 1.2 *"How many pairs of rabbits will be produced in a year, beginning with a single pair of matured rabbits, if every month each pair produced a new pair, which becomes productive two months after birth?"*

Solution. Let's sketch a picture to give us an idea of the problem.

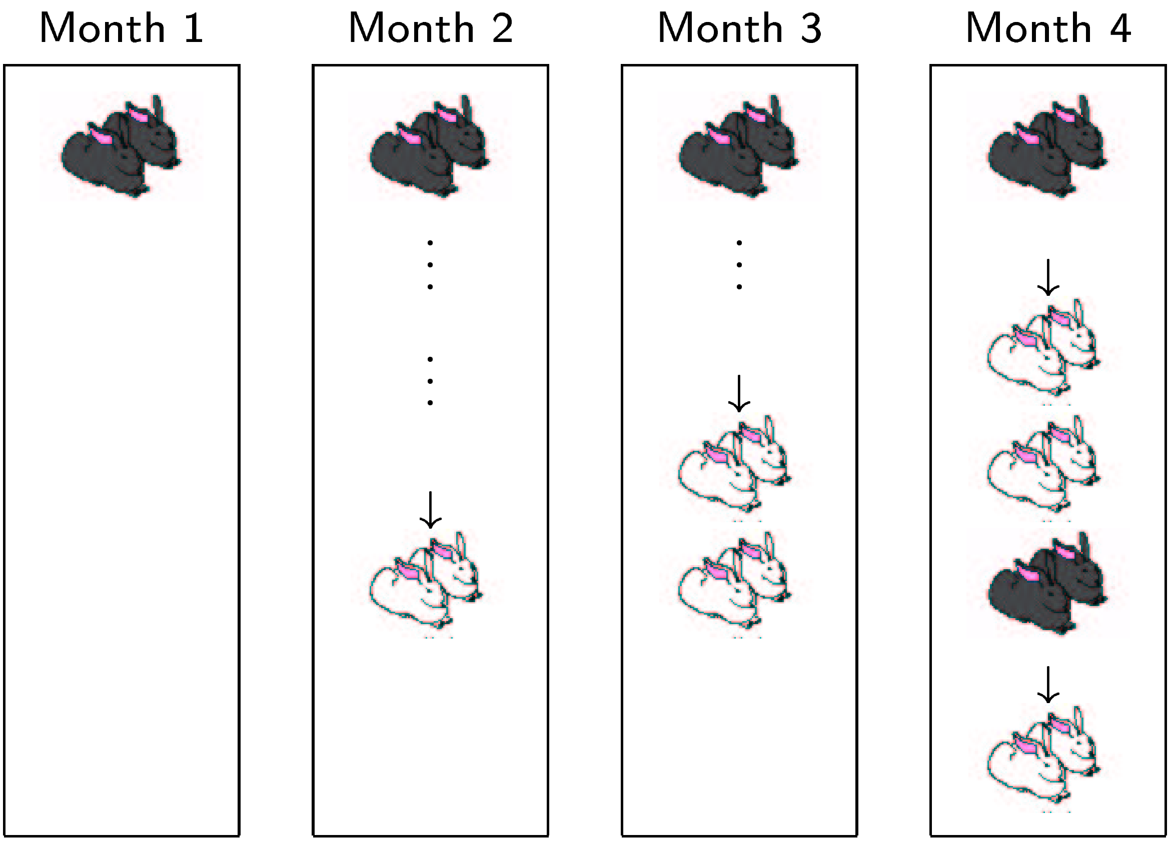

Figure 1.1: A mature pair of rabbits (shaded grey) each month produce a new pair of rabbits (shaded white). The new rabbits mature after two months.

We have assumed that none of the rabbits die.

If we denote y_k to be the number of pairs of rabbits in month k, where k is an integer ≥ 1, then from the diagram we can deduce

$$\left.\begin{array}{lcl} y_1 & = & \underline{\hspace{2cm}} \\ y_2 & = & \underline{\hspace{2cm}} \\ y_3 & = & \underline{\hspace{2cm}} \\ y_4 & = & \underline{\hspace{2cm}} \\ & \vdots & \\ & \downarrow & \end{array}\right\}$$

This sequence defines what are normally called the Fibonacci numbers.

To determine the number of pairs of rabbits at the end of the year we can continue calculating the number of rabbits month by month until we reach y_{12}.

Question 1.2 *Can we represent the Fibonacci numbers as a difference equation?*

The key do answering a question like this is to first write a **word equation** in which on the right hand side of the equation we write down the change in the 'quantity' whilst on the left hand we write down the reason why the 'quantity' is changing.

$$\left\{ \frac{\text{change in}}{\text{quantity}} \right\} = \left\{ \frac{\text{reasons why}}{\text{the quantity changed}} \right\}.$$

The change in the quantity will *always* be given by

$$\left\{ \begin{array}{c} \text{change in} \\ \text{quantity} \end{array} \right\} = \left\{ \begin{array}{c} \underline{\text{current value}} \\ \text{of quantity} \end{array} \right\} - \left\{ \begin{array}{c} \underline{\text{previous value}} \\ \text{of quantity} \end{array} \right\}.$$

Thus, for any problem of this type, we can write

$$\left\{ \begin{array}{c} \underline{\text{current value}} \\ \text{of quantity} \end{array} \right\} - \left\{ \begin{array}{c} \underline{\text{previous value}} \\ \text{of quantity} \end{array} \right\} = \left\{ \begin{array}{c} \underline{\text{reasons why the}} \\ \text{quantity changed} \end{array} \right\}.$$

For the rabbit problem this is

$$\left\{\begin{array}{c}\text{number of rabbits}\\ \text{present this month}\end{array}\right\} - \left\{\begin{array}{c}\text{number of rabbits}\\ \text{present last month}\end{array}\right\}$$
$$= \left\{\begin{array}{c}\text{reasons why the}\\ \text{number of rabbits changed}\end{array}\right\}. \quad (1.4)$$

Since we neglect rabbit's death, their total number can only be affected by births.

$$\left\{ \begin{array}{c} \text{reasons why the} \\ \text{number of rabbits changed} \end{array} \right\} = \left\{ \begin{array}{c} \underline{\text{number born}} \\ \text{this month} \end{array} \right\}$$

As the rabbits take two months to become productive and then produce only one pair per month, the last term on the RHS can be written as

$$\left\{ \begin{array}{c} \text{number born} \\ \text{this month} \end{array} \right\} = \left\{ \begin{array}{c} \dfrac{\text{number present}}{\underline{\text{two months ago}}} \end{array} \right\} \qquad (1.5)$$

The above "word equation" is true provided the current month is at least the third month. Combining the word equations (1.4) & (1.5) we obtain

$$\left\{ \begin{array}{c} \text{number present} \\ \text{this month} \end{array} \right\} = \left\{ \begin{array}{c} \text{number present} \\ \underline{\text{last month}} \end{array} \right\} + \left\{ \begin{array}{c} \text{number present} \\ \underline{\text{two months ago}} \end{array} \right\}.$$

which is a relationship between the number of pairs this month and the number of pairs in the previous two months.

If we denote the present month by k $(k \geq 3)$, then the previous month was the _____th month and the month before that was the _____th month.

Hence using the above word equation we can write down the difference equation

_________________ where $k = 3, 4, 5, \ldots$.

The above is often referred to as the Fibonacci equation.

To solve the above difference equation we undertake a process of iteration. But before we begin we need some initial conditions. In this case we need y_1 and y_2.

$$y_1 = 1,$$
$$y_2 = 2.$$

Putting $k = 3$ we get

$$y_3 = \underline{\hspace{5cm}}$$
$$= \underline{\hspace{5cm}}$$

Putting $k = 4$ we get

$$y_4 = \underline{\hspace{5cm}}$$
$$= \underline{\hspace{5cm}}$$

Question 1.3 *By repeating the above process find the pairs of rabbits at the end of the year (i.e. find y_{12}).*

I hope that you are becoming comfortable with the idea of discrete systems and how easy it is to find their solutions using iteration.

1.4 Formal Definitions

Definition 1.1 (Difference Equation) *A difference equation may be seen as a rule which expresses each member of a sequence, from some point on, in terms of previous members of the sequence*

Definition 1.2 (State) *If n takes on integer values, then x_n is called the state of the system at time n.*

Definition 1.3 (Order) *If we denote P and Q to be the largest and smallest subscripts on the variable that occur in the difference equation respectively, then the order of the difference equation is given by $P - Q$.*

Example 1.3 *Find the order of the following difference equations:*

(a) $x_{n+1} - 2x_n = 1$

(b) $x_n + x_{n-1} = e^{x_{n-1}}$

(c) $x_{n+2} - 2x_{n+1} = \frac{1}{x_{n-1}}$

Definition 1.4 ***(Order and Initial Conditions)***
To solve a difference equation of order $\underline{m}$, there must be $\underline{m}$ initial conditions.

Question 1.4 *The Fibonacci equation is given by*

$$y_k = y_{k-1} + y_{k-2}$$

1. *What is the order of the Fibonacci equation?*
2. *How many initial conditions do we need to solve the Fibonacci equation?*

Definition 1.5 ***(Linear and non-linear difference equations)*** *A difference equation of order n is said to be* <u>*linear*</u> *if it can be written in the form*

$$x_{k+n} + a_1(k)\, x_{k+n-1} + a_2(k)\, x_{k+n-2} + \dots + a_{n-1}(k)\, x_{k+1} + a_n(k)\, x_k = R_k,$$

where $a_k(k)$, $i = 1, 2, \dots, n$ and R_k are given functions of k. Otherwise it is termed <u>*non-linear*</u>*.*

Example 1.4 *State whether the following difference equations are linear or non-linear. Justify your answer.*

(a) $x_{n+1} - 2x_n = x_{n-1}$.

(b) $x_{n+1} + x_n = e^{x_n}$.

(c) $x_{n+1} - 2nx_{n-1} = n^2$.

(d) $x_{n+2} - 2x_{n+1} = \frac{1}{x_{n-1}}$.

Definition 1.6 (Solution) *A solution of a difference equation is an expression for* x_n *as a function of* n *such that when the solution is substituted into the difference equation, it satisfies this equation for all* n *which are specified.*

To show that the function $x_n = g(n)$ is a solution of a given difference equation we must do two things:

(1) Check that the solution satisfies any initial conditions that are given.

(2) Substitute the solution into the difference equation and show that the left hand side of the equation reduces to the right hand side of the equation.

Example 1.5 *Show that* $x_n = \frac{1}{2}n(n+1)$ *is a solution of the difference equation* $x_n - x_{n-1} = n$.

Solution We do not need to check that the solution satisfies the initial condition, as no initial condition is specified. Now note that if
$x_n = \frac{1}{2}n(n+1)$ then
$x_{n-1} = \underline{\hspace{5cm}}$.

Substituting these into the LHS of the difference equation yields

LHS =

= RHS

[Remember not to be in a hurry to expand things out.]

Hence $x_n = \frac{1}{2}n(n+1)$ is a solution of the difference equation.

All we are doing here is to substitute the form of x_n into our difference equation to verify that the LHS=RHS. With these sorts of problems, begin with one side and show that the expression is equivalent to the other side by algebraic manipulation.

Example 1.6 *Verify that* $x_n = \frac{1}{c+n}$ *(where* c *is a constant) is a solution of the difference equation*

$$x_{n+1}(1 + x_n) = x_n \ .$$

Solution We do not need to check that the solution satisfies the initial condition, as no initial condition is specified. Now note that if $x_n = \frac{1}{c+n}$ then $x_{n+1} =$ ____________________. Substituting into the LHS of the difference equation.

$$\Rightarrow \quad \text{LHS} =$$

$$= \text{RHS}$$

Hence $x_n = \frac{1}{c+n}$ is indeed a solution of the given difference equation.

Although iteration has the advantage of being easy to apply, it has one major drawback. For example, if one wishes to calculate y_{1000} by iteration, one is required to work out $y_1, y_2, \ldots, y_{999}$ regardless of whether you were interested in these values.

It would be useful to come up with a simple formula, where possible. Such a formula, if it exists, is said to provide a **“closed-form”** solution of the difference equation and would allow y_{1000} to be calculated directly without the need to work out all the preceding members of the sequence.

For instance, $x_n = \frac{1}{c+n}$ is the closed-form solution to example 1.6. We can quickly calculate that $x_{1000} = \frac{1}{c+1000}$.

Example 1.7 *Guess a "closed-form" solution to the difference equation*

$$y_k = 1 + y_{k-1} + 2\sqrt{1 + y_{k-1}} \quad k = 2, 3, \ldots$$

with $y_1 = 0$.

Solution As in example 1.2, it is best to work out a few members of the sequence to get an idea of any patterns that may exist

$$y_1 = 0, \ y_2 = 3, \ y_3 = 8, \ y_4 = 15$$

Notice anything? Any patterns emerging here?

This leads us to **guess**

$\underline{\hspace{6cm}}$, $k \geq 1$

At this stage our expression still remains as a guess as it has not been proven for all $k \geq 1$.

To verify that this is a solution of the difference equation, we must first check that the initial condition is satisfied: $y_1 = \underline{1^2 - 1 = 0}$. The initial condition *is* satisfied. We now substitute the above into the RHS of the difference equation and see if it equals the LHS, as was done previously.

$$\begin{aligned} \text{If} \quad y_k &= \\ \text{then} \quad y_{k-1} &= \\ \text{RHS} &= \end{aligned}$$

Don't be in a hurry to expand!!

$$= \text{LHS}$$

Therefore our guess of $y_k = \underline{k^2 - 1}$ is indeed a solution of the difference equation, as it satisfies the equation **and** the initial condition is also satisfied.

Definition 1.7 ***(Autonomous and non-autonomous)*** *If the subscript of a difference equation present explicitly in the equation, i.e. not just in the subscript of a state variable, then we refer to it as an* <u>*non-autonomous*</u> *difference equation. Otherwise it is called* <u>*autonomous*</u>*.*

Question 1.5 *Which of the following difference equations autonomous? (Give reasons.)*

(a) $y_{n+2} = y_{n+1} + y_n + n^2$

(b) $x_{n+1} - nx_n = 1$

(c) $x_{k+2} + x_{k+1} = \frac{1}{x_k}$

1.5 Concept map

Draw a concept map for this chapter relating the aims/key ideas of the chapter. If you are unfamiliar with the idea of a concept map see appendix A.

1.6 Questions

1. Give the orders of the following difference equations and state if they are linear or not and whether the equation is autonomous or non-autonomous.
   (i) $x_{n+1} + 2x_{n-1} = \frac{1}{n}$,
   (ii) $x_{n+3} + \frac{1}{x_{n+2}} = x_{n+2}$,
   (iii) $x_{n+4} + 2x_{n-1} = (n-2)^3$.
   (iv) $y_{n+2} = y_{n+1}y_n$.
   (v) $y_{n+1} = 2y_n + \sin(n)$.
   (vi) $n_{k+2} = n_{k+1} - n_k$.
2. Show that
   (a) $x_n = n(n+2)$ is a solution of the difference equation

$$x_{n+1} - x_{n-1} = 4(n+1)$$

(b) $x_n = n^2 + 2n - 1$ is a solution of the difference equation

$$x_n - x_{n-1} = 2n + 1, \quad x_1 = 2.$$

(c) (i) Show that $x_n = an + b$ is a solution of the difference equation

$$x_{n+1} - 2x_n + x_{n-1} = 0,$$

where a and b are constants.

(ii) Find the solution of the difference equation

$$n_{x+1} - 2n_x + n_{x-1} = 0, \qquad n\,(x = 1)$$

3. For the difference equation

$$y_k = 2y_{k-1}, \quad k = 2, 3, 4, \ldots$$

and initial condition $y_1 = 1$

(i) calculate y_2, y_3, y_4, y_5 and make a guess at the "closed-form" solution of y_k,

(ii) verify that your formula satisfies the difference equation and the initial condition.

4. Consider the difference equation

$$y_k = k y_{k-1}, \quad k = 1, 2, 3 \ldots$$

with initial condition $y_0 = 1$.

(a) Calculate y_1, y_2, y_3, y_4 and make a guess at the "closed-form" solution of y_k.

(b) Verify that your formula satisfies the difference equation and the initial condition.