

3 Applications of First-Order Difference Equations: Finance

3.1 Aims

In this chapter we will derive first-order difference equations for some common financial problems. We will do this by applying the principles from chapter 1 that

$$\left\{ \begin{array}{l} \text{change in} \\ \text{quantity} \end{array} \right\} = \left\{ \begin{array}{l} \text{reasons why the} \\ \text{quantity changed} \end{array} \right\}.$$

and that

$$\left\{ \begin{array}{l} \text{change in} \\ \text{quantity} \end{array} \right\} = \left\{ \begin{array}{l} \text{current value} \\ \text{of quantity} \end{array} \right\} - \left\{ \begin{array}{l} \text{previous value} \\ \text{of quantity} \end{array} \right\}.$$

The equations that we derive will be of the form

$$x_n - ax_{n-1} = b(n).$$

The solution of such an equation can be found using the general solution that we found in the previous chapter,

$$x_n = x_0 a^n + \sum_{p=1}^{p=n} a^{n-p} b(p). \quad (3.1)$$

After working through this chapter you should be able to

1. Set up a difference equation relevant to simple finance problems (i.e., write down the word equation, define the variables and obtain the appropriate difference equation).
2. Find the solution of your difference equation using equation 3.1.
3. Use your solution to answer simple financial problems.

3.2 Interest

A sum of money placed in a bank or financial institution earns interest according to the following relationship

$$\left\{ \begin{array}{l} \text{Amount in Bank} \\ \text{after } n \text{ periods} \end{array} \right\} - \left\{ \begin{array}{l} \text{Amount in Bank} \\ \text{after } (n - 1) \text{ periods} \end{array} \right\} = \left\{ \frac{\text{interest}}{\quad} \right\}$$

There are essentially two types of interest: simple or compound.

3.2.1 Simple Interest

3.2.1.1 The simple interest formula

The interest earned in any year is obtained by applying the rate to the *initial principal*, thus the investment remains the same throughout the given period. Hence

$$\left\{ \begin{array}{l} \text{Total amount} \\ \text{in Bank after} \\ n \text{ years} \end{array} \right\} - \left\{ \begin{array}{l} \text{amount} \\ \text{deposited} \\ \text{after } (n - 1) \\ \text{years} \end{array} \right\} = \left\{ \text{interest} \right\} \tag{3.2}$$

If the bank gives an annual interest of $p\%$, then the interest earned during any one year is

$$\left\{ \text{interest} \right\} = \frac{p}{100} \times \left\{ \text{principal} \right\}.$$

S_k is the total amount in the bank after n years, then equation (3.2) becomes

$$S_k = S_{k-1} + \left(\frac{P}{100} \right) S_0$$

Solving this gives:-

$$\begin{aligned} S_k &= S_0 + \sum_{r=1}^k \frac{p}{100} S_0 \\ &= S_0 + \frac{kp}{100} S_0 \\ &= \left(1 + \frac{kp}{100} \right) S_0 \end{aligned}$$

which is called the simple interest formula.

In questions when time is given in *days*, we may calculate either *exact simple interest*, on the basis of a 365-day year (leap year or not) or *ordinary simple interest*, on the basis of a 360-day year.

Ordinary simple interest brings increased revenue to the lender. The general practice in the United States and in international business transactions is to use ordinary simple interest. Unless otherwise specified all questions using days are to be answered using ordinary simple interest.

Example 3.1 Find the interest on a 60-day loan of \$1500 at 15%. (\$36.25)

[Zimma & Brown.]

Solution The total amount of money owed at the end of the loan is given by

$$S_1 = \left(1 + \frac{kp}{100}\right) S_0, \text{ where}$$

$$k = \frac{1}{6}, p = 14.5, S_0 = 1500. \text{ Hence}$$

$$\begin{aligned} S_1 &= \left(1 + \frac{14.5}{600}\right) 1500, \\ &= \$1536.25. \end{aligned}$$

Thus the interest on the loan is $1536.25 - 1500.00 = \$36.25$.

3.2.1.2 Application of the simple interest formula To encourage prompt payment of invoices, manufactures and wholesalers offer discounts for payments in advance of the due date. The following typical terms may be printed on the sales invoice:

5/10, n/30.

Goods billed on this basis are subject to a discount of 5% if paid for within ten days from the date of invoice.

Otherwise, the full amount must be paid not later than 30 days from the date of the invoice.

Example 3.2 *Cyberdyne Systems*

Corporation receives an invoice for \$3000 with the above terms. They are experiencing a cash-flow problem and can not pay the invoice now, but they will be able to do so in thirty days time.

- (a) *What is the discount for paying within the first ten days?*
- (b) *Cyberdyne Systems Corporation decides to take a loan out to take advantage of the discount. Explain why they should take the loan out on day ten.*
- (c) *If Cyberdyne Systems Corporation take a loan out on day ten what loan do they need?*

$\$150$

$\$2850$

- (d) What profit can Cyberdyne Systems Corporation realize by borrowing money at 18% and paying the invoice on the 20th day from the date of invoice? $= \$12.50$
- (e) What is the highest interest rate at which they can borrow in order to take advantage of the discount? 94.73%
- (f) Why is it better to round down, rather than up, your answer to the previous question?

The largest possible interest rate is found by solving the equation

$$\left(1 + \frac{\frac{20}{360}p}{100}\right) 2850 = 3000 \quad \text{for } p,$$

$$\Rightarrow p = 94.73\%$$

3.2.2 Compound Interest

3.2.2.1 The compound interest formula

This type of interest is added to the principal at regular intervals, called conversion periods, and the new amount (rather than the principal) is used for calculating the interest for the next conversion period. The fraction of a year occupied by the conversion period is denoted by α so that conversion periods of 1 month, 6 months and 1 year are denoted by $\alpha = \frac{1}{12}$, $\alpha = \frac{1}{2}$ and $\alpha = 1$ respectively. The usual way of saying that the conversion period is one month is, "... the interest is compounded monthly."

For an interest rate of $p\%$ and a conversion period equal to a fraction α of a year, the interest earned for the period is $\alpha p\%$ of the amount on deposit at the start of the period, i.e.

$$\left\{ \text{interest} \right\} = \frac{\alpha p}{100} \times \left\{ \begin{array}{l} \text{amount on deposit} \\ \text{at the start of the} \\ \text{conversion period} \end{array} \right\}$$

If S_n is the amount in the bank after n conversion periods, then

$$\left\{ \begin{array}{l} \text{amount on deposit} \\ \text{after } n \\ \text{conversion periods} \end{array} \right\} - \left\{ \begin{array}{l} \text{amount on deposit} \\ \text{after } (n-1) \\ \text{conversion periods} \end{array} \right\} \\ = \frac{\alpha p}{100} \left\{ \begin{array}{l} \text{amount on deposit} \\ \text{after } (n-1) \\ \text{conversion periods} \end{array} \right\}$$

i.e. the corresponding difference equation:

$$S_n =$$

Solving it gives

which is called the compound interest formula.

Note

1. n denotes the number of *conversion periods* (rather than the number of years as was the case for the Simple Interest formula). Thus to find the amount in the bank after **5 years** at compound interest with a **monthly** conversion period, we should take $n = 60$ in the above formula [taking $n = 5$ will only give us the amount after 5 months in this case!!].
2. The simple interest formula is linear, but the compound interest formula is exponential.

Example 3.3 Find the interest earned on \$1000 invested for two years at 12% compounded semiannually. (\$262.48)
[Zimma & Brown.]

Solution The total amount of money at the end of the investment is given by

$$S_n = \left(1 + \frac{\alpha p}{100}\right)^n S_0, \text{ where}$$

$$\alpha = _, p = _, n = _,$$

$$S_0 = ______. \text{ Hence}$$

$$S_1 = ______,$$

$$= ______.$$

$$\frac{1}{2} \quad 12 \quad 4$$
$$1000$$

$$\left(1 + \frac{12}{200}\right)^4 1000$$
$$\$1262.48$$

Thus the interest earned on the investment is $1262.48 - 1000.00 = \$262.48$.

3.2.2 Application of the compound interest formula Two nominal rates of interest with different frequencies of conversion are said to be *equivalent* if they yield the same accumulated value at the end of one year (and hence, at the end of any number of years).

Example 3.4 *What is the annual effective compound interest rate for interest at 10% compounded semi-annually?*

Solution If you invest S_0 dollars into an account earning 10% compounded semi-annually then at the end of the year you have

$$S_2 =$$

dollars in your account. If your money were invested in an account paying $p\%$ compounded annually you would have

$$S_1 =$$

$$\left(1 + \frac{\frac{1}{2}10}{100}\right)^2 S_0$$

$$\left(1 + \frac{1 \cdot 10}{100}\right)^2 S_0.$$

Thus we need

$$\begin{aligned}\left(1 + \frac{1 \cdot p}{100}\right)^1 S_0 &= \left(1 + \frac{\frac{1}{2}10}{100}\right)^2 S_0, \\ \Rightarrow p &= 100 [(1.05)^2 - 1] \\ &= 10.25\%.\end{aligned}$$

Example 3.5 *What interest rate compounded quarterly is equivalent to 8% compounded semi-annually?*

Solution To find the equivalent interest rate, p , calculate the amount of money in both investments after one-year. Thus we have

$$\begin{aligned} \left(1 + \frac{\frac{1}{4}p}{100}\right)^4 &= \left(1 + \frac{\frac{1}{2}8}{100}\right)^2, \\ \Rightarrow 1 + \frac{p}{400} &= \sqrt{1 + 0.04}, \\ \Rightarrow p &= 400 \left(\sqrt{1.04} - 1\right), \\ \Rightarrow p &= 7.922\%. \end{aligned}$$

3.3 Loan Repayments (Amortization Scheme)

3.3.1 The loan repayment formula

The most common method of repaying an interest-bearing loan, such as for housing loans or mortgages, is by *amortization*. In this method, a series of payments (denoted R) are made at regular intervals, and usually in equal amounts, to reduce the principal (the amount borrowed) and to pay interest on the amount still owing. It is supposed that the compound interest idea is used here. Hence

$$\begin{aligned} \left\{ \begin{array}{l} \text{Debt after} \\ \text{n payments} \end{array} \right\} &- \left\{ \begin{array}{l} \text{Debt after} \\ (n-1) \text{ payments} \end{array} \right\} \\ &= \left\{ \begin{array}{l} \text{interest} \\ \text{on debts} \end{array} \right\} - \left\{ \text{payments} \right\} \end{aligned}$$

Let the outstanding debt after n payments be D_n and the repayment R . Then the above word equation can be written in the form of the following difference equation:

$$D_n =$$

$$=$$

Solving it gives:

which we call the loan repayment formula.

The common commercial practice is to round the payment up to the nearest cent. Rounding up the payment amount results in a reduced last payment, but we will ignore this here.

Suppose that we borrow an amount D_0 which we pay off after n regular payments. Interest is compounded at a rate $p\%$ at a frequency α . What is the size of the regular payment R ?

After the last payment we owe _____.

Thus _____ and we have

$$0 = \left(1 + \frac{\alpha p}{100}\right)^n \left(D_0 - \frac{100R}{\alpha p}\right) + \frac{100R}{\alpha p}, \quad (3.3)$$

Let

$$i = \frac{\alpha p}{100}.$$

Then re-arranging equation (3.3) we obtain

$$R = \quad (3.4)$$

$$\frac{(1+i)^n}{[(1+i)^n - 1]} \cdot iD_0.$$

Example 3.6 *A loan of \$10,000 is to be repaid according to the amortization scheme. Calculate the monthly repayments needed to pay off the loan in 5 years if interest is charged at 15% compounded monthly.*

Solution

Let R be the required monthly payment. In equation (3.4) choose

$$\alpha = \quad \text{why?}$$

$$p =$$

$$D_0 =$$

$$n = \quad \text{why?}$$

and we obtain

$$R = \frac{(1+i)^{60}}{[(1+i)^{60} - 1]} \cdot i(10000),$$

$$i = \frac{\frac{1}{12} \times 15}{100},$$

$$\Rightarrow R = \$237.90.$$

Question 3.1 *What is the total repayment on the loan of \$10000?*

3.3.2 The outstanding principle and equity in a loan

Suppose that we take a loan from a bank and repay it according to an amortization scheme with n payments. After k payments ($k < n$) we will be in debt to the bank.

The indebtedness at any time is called the outstanding balance or the outstanding principle.

Let us return to the problem considered in example 3.6 in which a loan of \$10,000 is to be repaid according to the amortization scheme. The monthly repayments needed to pay off the loan in 5 years if interest is charged at 15% compounded monthly was found to be \$237.90.

At the end of the first month, before we make a repayment, we owe the bank

$$\left(1 + \frac{\frac{1}{12} \times 15}{100}\right) 10,000 = \$10125.$$

We pay the bank \$237.90. Of this money, \$125 is used to pay the interest on the unpaid balance. The remaining \$112.90 is used to reduce the outstanding principle to \$9887.10.

At the end of the second month, before we make a repayment, we owe the bank

$$\frac{\left(1 + \frac{1}{12} \times 15\right) 9987.10}{100} = \underline{\underline{\$10111.93875}} = \underline{\underline{\$10111.94}}.$$

(I have rounded the money up to make the following calculations a little easier). We pay the bank \$237.90. Of this money, \$111.94 used to pay the interest on the unpaid balance. The remaining \$125.60 is used to reduce the outstanding principle to \$9861.14.

Note that over the term of the loan the outstanding principle decreases.

Consequently the interest on the unpaid balance decreases and the amount we pay off the outstanding principle increases.

The amortization method is quite often used to pay off loans incurred in purchasing a property. In such cases, the outstanding principle is called the *seller's equity*. The amount of principal that has been paid already, plus the down payment, is called the *buyer's equity* or the *owner's equity*.

Clearly

buyer's equity + seller's equity = selling price.

Let us again return to example 3.6 and assume that the loan is for a property and that there was no down payment.

The the buyer's equity after one payment is \$112.90. The buyer's equity after the second payment is \$238.86 (\$112.90 + \$125.96).

Questions relating to the outstanding principle and the lendee's equity in a loan can be found in the miscellaneous questions section of this chapter.

3.3.3 Steven buys a car

Question 3.2 *Steven decides to purchase a car for \$40 00. He has savings of \$17 000 and has the choice of two payment schemes.*

- *He can put down a deposit of \$17 000 and take out a five-year loan (amortization scheme) from the bank with interest at 7.5% p.a. compounded quarterly.*
- *He can put down a deposit of \$15 000 and make weekly payments of \$105 for five years to the dealer. At the end of five years he makes a final payment of \$3500*

Question 3.2 *1. Which option should Steven choose (justify your answer)? How much money does he save?*

- 2. Steven opts to pay the dealer directly rather than take a loan out from the bank. He decides to invest the remaining \$2 000 of his savings in a five-year term deposit account with his bank. If interest is compounded annually what is the minimum interest rate that is required for his decision to make sense?*

Solution

1. In **option 1** $R = \$1380$. The total repayments are \$27794.00 and the cost of the car is \$44794.00. In **option 2** the cost of the car is \$45800.

Steven should choose option one as he will save \$1006.

2. Steven's decision only makes financial sense if the interest earned on his money is at least \$1006. The minimum interest rate is 8.49%.

3.4 Annuities

The term annuity has several meanings, but for our purposes we shall take it as a bank account into which equal sums (denoted I) are deposited at regular intervals. The money draws interest and accumulates for a certain number of years, after which it becomes available to the investor.

The time between successive payments of an annuity is called the *payment interval*. We assume that payments are made at the *end* of the payment interval. Such an annuity scheme is called an *ordinary annuity*. We assume that the payment interval and the interest conversion period coincide, such an annuity is known as a *simple annuity*.

The word equation describing the annuity scenario is:

$$\begin{aligned}
 & \left\{ \begin{array}{l} \text{Investment after} \\ n \text{ payments} \end{array} \right\} - \left\{ \begin{array}{l} \text{Investment after} \\ n - 1 \text{ payments} \end{array} \right\} \\
 = & \left\{ \begin{array}{l} \text{payment} \\ \text{into annuity} \end{array} \right\} + \left\{ \begin{array}{l} \text{interest earned on investment} \\ \text{after } n - 1 \text{ payments} \end{array} \right\}
 \end{aligned}$$

Hence by letting y_n be the amount of money in the annuity account after n time periods, the difference equation is:-

$$\begin{aligned}
 y_n - y_{n-1} &= \\
 &=
 \end{aligned}$$

Solving this gives:

$$\begin{aligned} & I + \left(\frac{\alpha p}{100}\right) y_{n-1} \\ \Rightarrow y_n & \left(1 + \frac{\alpha p}{100}\right) y_{n-1} + I \end{aligned}$$

$$y_n = \left(1 + \frac{\alpha p}{100}\right)^n \left(y_0 + \frac{100I}{\alpha p}\right) - \frac{100I}{\alpha p}.$$

Example 3.7 *Suppose that an annuity account is opened with \$20. Thereafter \$20 is deposited into it at the end of every quarter-year. Interest is earned at the annual rate of 8% compounded quarterly. How much money is in the annuity account after 10 years?*

$$y_0 = 20, \quad D = 20,$$

$$\alpha = \frac{1}{4}, \quad p = 1.08,$$

$$n = 40,$$

$$20 \qquad 20$$

$$\frac{1}{4} \qquad 8$$

$$40$$

$$\begin{aligned}y_{40} &= \left(1 + \frac{1}{50}\right)^{40} \left(20 + \frac{2000}{2}\right) - \frac{2000}{2} \\ &= \$1252.\end{aligned}$$

3.5 Revision of key ideas

3.6 Concept map

Draw a concept map for this chapter relating the aims/key ideas of the chapter. If you are unfamiliar with the idea of a concept map see appendix A.