12 Linear Stability Analysis of Steady-State Solutions

12.1 Physical motivation of 'stability'

Consider the differential equation

$$\frac{\mathrm{d}x}{\mathrm{d}t} = f(x). \qquad (12.1)$$

The steady-state solutions, x^*, of equation (12.1) are the solutions of

$$f(x^*) = \underline{0}.$$

At a steady-state the system does not change with time. (hyW7)

By carefuly examining what happens to a system when it is *near* a steady-state we can gain an insight into the dynamic behaviour of equation (5.1).

To do this we distinguish between two types of steady-states by re-introducing the the concept of *stability*.

Since this is best described by analogy, see figure 12.1, which exemplifies three situations, two of which are steady states; one steady state is stable, the other unstable.

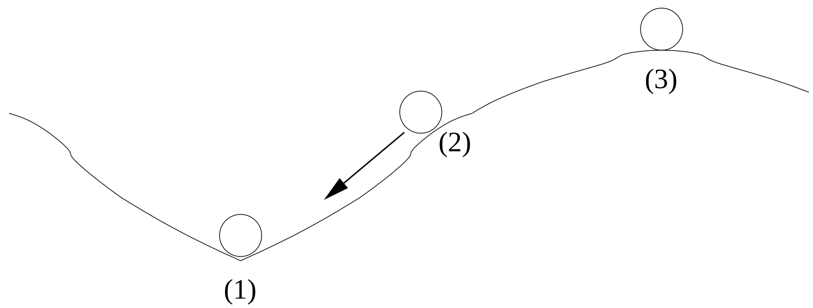

Figure 12.1: Balls one and three are at rest and represent steady-state situations. Ball one is stable; if moved slightly it will return to its former position. Ball three is unstable. The slightest disturbance will cause it to fall into one of the adjoining valeys. Ball two is not in a steady state, since its position and speed are continuously changing. (After Edelstein-Keshet, 1988)

A steady state is termed *stable* if neighbouring states are attracted to it and *unstable* if the converse is true.

As shown in figure 12.1, whilst an object balanced precariously on a hill may be in steady state, it will not return to this position if disturbed slightly. Rather, it may proceed on some lengthy excursion leading possibly to a second, more stable situation. (Graphical Technique?)

Such distinctions have *practical implications* in applied mathematics.

When steady states are unstable, great changes may be about to happen: a population may crash, possibly becoming extinct, or a disease may spread through a population.

This *qualitative* information about whether change is imminent is potential of *great importance*.

With this motivation behind us, we turn to the analysis that permits us to make such predictions.

12.2 Linear stability analysis

Suppose that we have already found a steady-state solution x^* of equation (12.1).

We now proceed to explore its stability by asking the following key question: given an initial condition $x(0)$ *close* to x^*, will the solution of the differential equation, i.e. the values $x(t)$, tend toward or away from the steady state x^*?

To answer this question we write

$$\xi(t) = \underline{x(t) - x^*}, \qquad \underline{|\xi(t)| \ll 1},$$

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where $\xi(t)$ is the distance between the population size at time t and the steady-state. If $\xi(t) > 0$ then $x(t) > x^*$ and conversely if $\xi(t) < 0$ then $x(t) < x^*$. Using equation (12.1) we can obtain a differential equation for $\xi(t)$.

$$\frac{d\xi}{dt} = \frac{d}{dt}\left[x(t) - x^*\right]$$

$$= \frac{dx}{dt} \qquad \text{(why?)}$$

$$= f(x^* + \xi) \quad \text{(why?)}$$

$) + f'(x^*)\,\xi + O(\xi^2)\,;$ (why?)

1.

* (why?)

______ . (12.2)

Thus we have obtained a *linear* differential equation for the distance between the solution of our differential equation, $x(t)$, and the steady-state x^*. In this equation the parameter λ is known as the *eigenvalue* of the steady-state x^*.

The solution of the differential equation

$$\frac{d\xi}{dt} = \lambda\xi$$

is

$$\xi(t) = \_\_\_\_\_\_\_\_\_\_\_\_,$$

$$\Rightarrow \lim_{t\to\infty} \xi(t) = \begin{cases} \_\_\_\_ & \text{if } \lambda < 0 \\ \_\_\_\_ & \text{if } \lambda > 0 \end{cases}$$

Thus the eigenvalue determines the stability of the steady-state.

Definition 12.1 ***(Stable and unstable fixed point)*** *A steady-state x^* is*

$$\textit{stable} \quad \textit{if} \quad f'(x^*) < 0 \qquad (12.3)$$

$$\textit{unstable} \quad \textit{if} \quad f'(x^*) > 0 \qquad (12.4)$$

The stability of the steady-state solution x^* is determined by the value of $f'(x^*)$.

Question 12.1 *Why does $\lambda < 0$ mean that the steady-state solution x^* is stable?*

Question 12.2 *Is the distinction between a stable steady-state solution and an unstable steady-state solution of practical importance?*

Comment 12.1 *If you start near a stable steady-state how quickly do you approach it?*

Suppose that x^* is a *stable* steady-state solution of the differential equation

$$\frac{\mathrm{d}x}{\mathrm{d}t} = f(x).$$

Then the stability of the steady-state can change in only one way: the eigenvalue $(\lambda = f'(x^*))$ *increases*/*decreases* through 0.

Question 12.3 *How many ways can the fixed-point x^* of the difference equation*

$$x_{n+1} = f(x_n)$$

lose stability?

12.3 Stability of the steady-states in the logistic equation

For the logistic differential equation we have

$$\frac{\mathrm{d}x}{\mathrm{d}t} = rx\left(1 - \frac{x}{K}\right). \tag{12.5}$$

Thus

$$f(x) = rx\left(1 - \frac{x}{K}\right)$$

and

$$\frac{\mathrm{d}f(x)}{\mathrm{d}x} = \underline{\qquad\qquad\qquad}.$$

$$\boxed{\frac{\mathrm{d}f(x)}{\mathrm{d}x} = r\left(1 - \frac{2x}{K}\right)}$$

The steady-states of the logistic equation are given by

$$\begin{aligned} f(x) &= 0, \\ rx\left(1 - \frac{x}{K}\right) &= 0, \\ x_1^* = 0 \quad &\text{or} \quad x_2^* = \underline{\quad}. \end{aligned}$$

The steady-states and their corresponding eigenvalues λ are

$$x_1^* = 0, \qquad \lambda = f'(x_1^*) = \underline{\qquad}, \tag{12.6}$$

$$x_2^* = K, \qquad \lambda = f'(x_2^*) = \underline{\qquad}. \tag{12.7}$$

$$x_1^* = 0, \qquad \lambda = f'(0) \ = r,$$

$$x_2^* = K, \qquad \lambda = f'(K) = -r.$$

For $r > 0$ both of these steady-states are biologically meaningful, that is non-negative. With $r > 0$ the trivial fixed point is _______ whereas the non-trivial steady-state is _____.

12.4 Stability of the steady-state in the Gompertz model

For the Gompertz model we have

$$\frac{\mathrm{d}x}{\mathrm{d}t} = rx \ln \frac{K}{x}$$

Thus

$$f\left(x\right) = rx \ln \frac{K}{x}$$

and

$$\frac{\mathrm{d}f\left(x\right)}{\mathrm{d}x} = \underline{\qquad\qquad}$$

The steady-state of the Gompertz model is give by

$$f(x) = 0,$$
$$rx \ln \frac{K}{x} = 0,$$
$$x^* = \_$$

The corresponding eigenvalue is given by

$$\lambda = f'(x^*) = \_\_\_\_\_.$$

For $r > 0$ the steady-state solution x^* is therefore _____.

12.5 Steady-state diagrams and bifurcations

Many problems of practical interest contain a control, or bifurcation, parameter λ. Instead of writing

$$\frac{\mathrm{d}x}{\mathrm{d}t} = f(x),$$

such problems are written in the form

$$\frac{\mathrm{d}x}{\mathrm{d}t} = f(x, \lambda). \tag{12.8}$$

The steady-state solutions are found by solving the equation

$$f(x, \lambda) = 0.$$

In such problems the value of any steady-state solutions and their stability is usually a function of the control parameter. This information is useful expressed in a steady-state, or response, diagram.

Definition 12.2 ***Steady-state diagram***
The graph of x versus λ is called a steady-state diagram or a response curve. This shows how the steady-state solutions of equation (12.8), x, depend upon the control (bifurcation) parameter λ.

In such a diagram steady-states are indicated by using a solid line whilst unstable steady-states are indicated by a dotted line. See figures 12.2–12.4 for examples.

Of particular interest on steady-state diagrams are *bifurcation points.* Loosely speaking, a bifurcation point is a point (x, λ) on a steady-state diagram where two solution branches with distinct tangents intersect. At such a point the number of steady-state solutions to the differential equation changes.

Definition 12.3 (Bifurcation point)
The point (x, λ) is called a bifurcation point if the number of steady-state solutions of equation (12.8) in the neighbourhood of x is not constant for any arbitrary small change of λ.

In sections 12.5.1 12.5.3 we analyse three differential equations. The steady-state diagram for each equation contains a different type of bifurcation point.

12.5.1 The limit-point bifurcation

Consider the differential equation

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \mu - x^2. \qquad (12.9)$$

The set of steady-state solutions is given by

$$x = \underline{\pm\sqrt{\mu}}, \qquad \mu \geq 0.$$

There are *no* steady-state solutions when $\mu < 0$, one steady-state solution when $\mu = 0$ and two steady-state solutions when $\mu > 0$.

The point $(\mu = 0, x = 0)$ is therefore a *bifurcation point* and the value $\mu = 0$ is a *bifurcation value.*

Question 12.4 *Show that the steady-state $x = +\sqrt{\mu}$ is stable ($\mu > 0$) whilst the steady-state $x = -\sqrt{\mu}$ is unstable ($\mu > 0$).*

The steady-state diagram for equation (12.9) is shown in figure 12.2.

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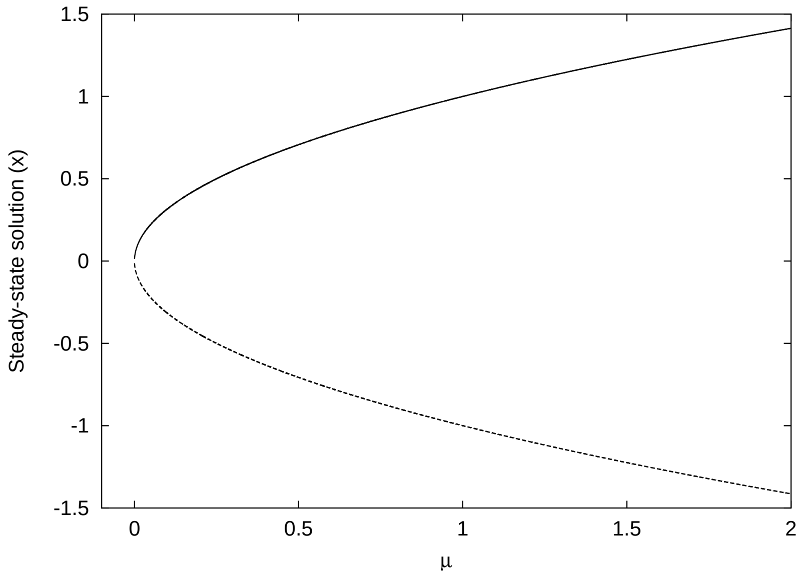

Figure 12.2: Steady-state diagram for the differential equation $\frac{\mathrm{d}x}{\mathrm{d}t} = \mu - x^2$: a solid line indicates a stable solution whilst a dotted line indicates an unstable solution.

The particular type of bifurcation occurring in figure 12.2 (i.e., where on one side of a parameter value there are *no* fixed points and on the other side there are *two* fixed points) is known as a *saddle-node bifurcation* or a *limit-point bifurcation*.

12.5.2 The transcritical bifurcation

Consider the differential equation

$$\frac{dx}{dt} = \mu x - x^2. \tag{110}$$

The set of steady-state solutions is given by

$$x_1 = 0,$$
$$x_2 = \mu.$$

There are *two* steady-state solutions for $\mu \neq 0$ and one steady-state solution when $\mu = 0$. Thus the number of steady-state solutions changes at the point $(\mu, x) = (0, 0)$. Thus this point is a *bifurcation point.*

Question 12.5

1. *Show that the steady-state branch $x_1 = 0$ is stable for $\mu < 0$ and unstable for $\mu > 0$.*
2. *Show that the steady-state branch $x_2 = \mu$ is unstable for $\mu < 0$ and stable for $\mu > 0$.*

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The steady-state diagram for equation (2.110) is shown in figure 2.13. Note that at the point $(\mu, x) = (0, 0)$, which we have already identified as a bifurcation point, two solution branches with distinct tangents intersect.

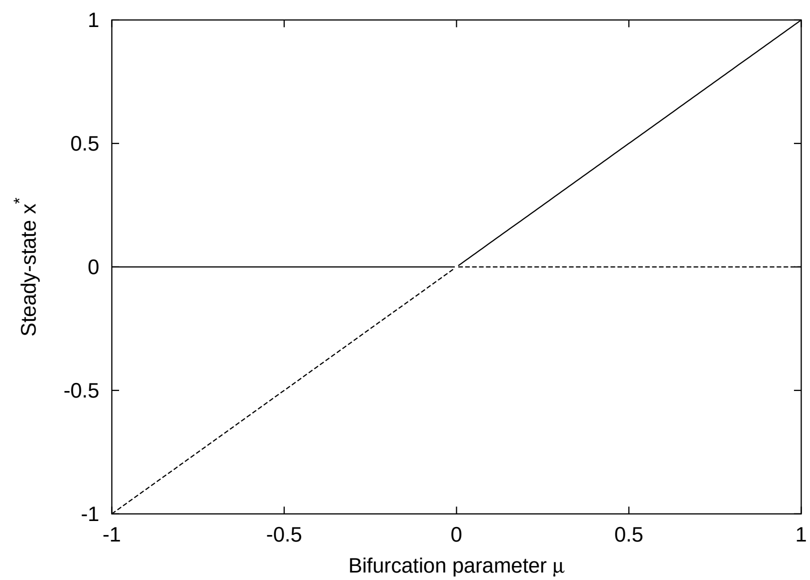

Figure 2.13: Steady-state diagram for the differential equation $\frac{dx}{dt} = \mu x - x^2$: a solid line indicates a stable solution whilst a dotted line indicates an unstable solution.

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The particular type of bifurcation occurring in figure 12.3 (i.e., where on one side of a parameter value there are *two* fixed points and on the other side there are *two* fixed points) is known as a *transcritical bifurcation*. This bifurcation frequently occurs in mathematical epidemiology.

Question 12.6 *Outline the reasons why a transcritical bifurcation is expected to occur in mathematical epidemiology.*

2.5.3 The pitchfork bifurcation

Consider the differential equation

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \mu x - x^3. \tag{2.11}$$

The set of steady-state solutions is given by

$$x_1 = 0,$$
$$x_{\pm} = \pm\sqrt{\mu}, \qquad \mu \geq 0.$$

There is *one* steady-state solutions when $\mu \leq 0$ and three steady-state solutions when $\mu > 0$. The number of steady-state solutions to the differential equation changes at the point $(\mu, x) = (0, 0)$. Thus this point is a *bifurcation point*.

Question 12.7

1. *Show that the steady-state* $x_1 = 0$ *is stable for* $\mu < 0$ *and unstable for* $\mu > 0$ *.*
2. *Show that the steady-state* $x_{\pm} = \pm\sqrt{\mu}$ *is stable for* $\mu > 0$*.*

The steady-state diagram for equation (2.11) is shown in figure 2.4. Note that at the point $(\mu, x) = (0, 0)$, which we have already identified as a bifurcation point, two solution branches with distinct tangents intersect.

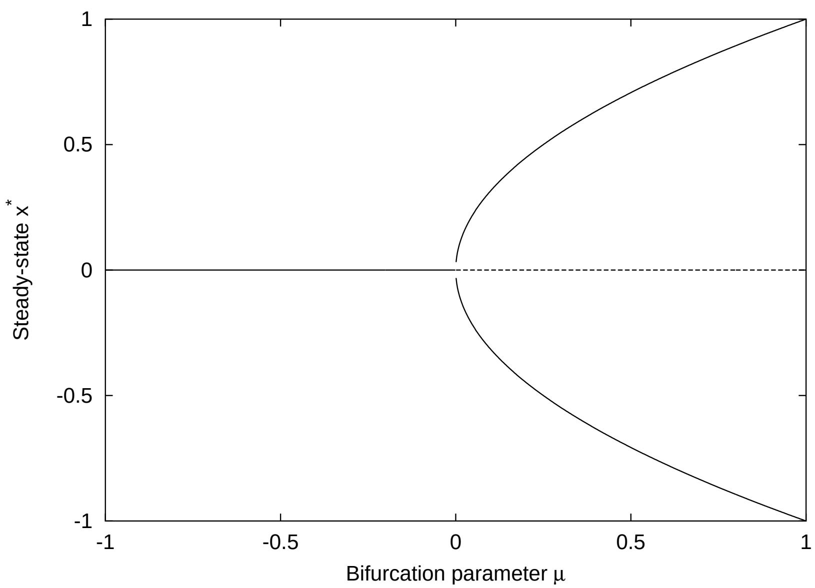

Figure 12.4: Steady-state diagram for the differential equation $\frac{dx}{dt} = \mu x - x^3$: a solid line indicates a stable solution whilst a dotted line indicates an unstable solution.

The particular type of bifurcation occurring in figure 12.4 (i.e., where on one side of a parameter value there is *one* fixed points and on the other side there are *three* fixed points) is known as a *pitchfork bifurcation*.

The study of steady-state solutions, their stability and their bifurcations is the process through which the properties of non-linear differential equations are elucidated in a systematic manner. These issues are left for another course.

1.6 Revision of key ideas

1.7 Concept map

Draw a concept map for this chapter relating the aims/key ideas of the chapter. If you are unfamiliar with the idea of a concept map see appendix A.