9 First-Order Differential Equations

9.1 Aims

In this chapter we focus on solving first-order differential equations. We can write a first-order differential equation in the general form

$$\frac{dy}{dx} = f(y, x).$$

We will consider three cases, corresponding to special cases of the function $f(y, x)$. These cases are:

The integrable case: The function $f(y, x)$ is independent of y, i.e.

$$f(y, x) = g(x).$$

The separable case: The function $f(y, x)$ can be writen in the form

$$f(y, x) = \frac{g(x)}{h(y)}$$

The integrating factor case: This applies when the function $f(y, x)$ can be written in the form

$$f(y, x) = -P(x)\, y + g(x).$$

After working through this chapter you should be able to:

1. solve first-order DEs that are:
   - integrable;
   - separable.
2. solve first-order DEs using the integrating factor method.
3. check that a given expression is indeed the solution of a given differential equation.

9.2 Solving a first-order differential equation: Integrable equations

The first-order differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = g(x)$$

has solution

$$y = \int g(x)\,\mathrm{d}x + c.$$

where c is an arbitrary constant.

Let us solve a differential equation of this type

Example 9.1 *Find the solution of*
$\frac{dy}{dx} - \sin x + x^2 = 0$

Solution

To solve $\frac{\mathrm{d}y}{\mathrm{d}x} - \sin x + x^2 = 0$ for $y(x)$ write

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \sin x - x^2$$
$$\Rightarrow y(x) = \underline{\int \left(\sin x - x^2\right) \mathrm{d}x}$$
$$= \underline{\qquad\qquad\qquad}$$

Question 9.1 *Check that* $y(x) = -\cos x - \frac{1}{3}x^3 + c$ *is the solution to example 9.1.*

$\left(\frac{dy}{dx} - \sin x + x^2 = 0\right)$

Example 9.2 *Find the solution of* $\frac{dy}{dx} = ax.$

Solution

$$y(x) = \_\_\_\_\_\_$$
$$= \_\_\_\_\_\_$$

where c is an arbitrary constant.

9.3 Solving a first-order differential equation: Separable equations

If

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{g(x)}{h(y)}$$

then

$$\underline{\int h(y)\,\mathrm{d}y} = \underline{\int g(x)\,\mathrm{d}y}.$$

Remember the constant of integration $+c$, when integrating.

Example 9.3 *Solve* $\frac{dy}{dx} = \frac{y}{1+x}$.

Solution

$$\int \frac{1}{y}\mathrm{d}y = \int \frac{1}{1+x}\mathrm{d}x$$

$$\Rightarrow \ln|y| = \underline{\qquad\qquad}$$

$$\Rightarrow y = \underline{\qquad\qquad}$$

$$= \underline{\qquad\qquad}$$

$$= k(1+x)$$

where $k = e^c$ is a constant.

Example 9.4 *Solve* $\frac{dy}{dx} = \frac{-x}{y}$ *with the initial condition* $y(1) = 1$.

Solution

$$\int y\mathrm{d}y = -\int x\mathrm{d}x$$

$$\Rightarrow \underline{\frac{1}{2}y^2} = \underline{\qquad\qquad}$$

where c is an arbitrary constant.
Applying the initial condition $y(1) = 1$ implies $c = \underline{1}$, giving the solution

$$\underline{\frac{1}{2}y^2} = \underline{\qquad\qquad}$$

$$y^2 + x^2 = \underline{\ }.$$

Question 9.2 *What object does the solution to example 9.4 describe?*

Example 9.5 *Solve* $\frac{dy}{dt} = -ay$ *with the initial condition* $y(0) = y_0 > 0$.

Solution

$$\int \frac{1}{y} \mathrm{d}y = -a \int \mathrm{d}t$$

giving the solution

$$y = y_0 e^{-at}.$$

Example 9.6 *Solve* $V \frac{dy}{dt} = q(y_{in} - y)$ *with the initial condition* $y(0) = y_0 > 0$ *subject to* $q > 0$ *and* $y_{in} > 0$.

Solution

$$\int \frac{1}{y_{\mathrm{in}} - y} \mathrm{d}y = -\frac{q}{V} \int \mathrm{d}t$$

with solution $y = y_{\mathrm{in}} - (y_{\mathrm{in}} - y_0) e^{-qt/V}$.

9.4 Solving a first-order linear differential equations: Integrating factors

To solve the general first-order linear differential equation

$$\frac{dy}{dx} + P(x)\,y = g(x)$$

calculate the integrating factor

$$R(x) = \underline{\exp\left[\int P(x)\,dx\right]}$$

then multiplying both sides of the equation by the integrating factor gives

$$R(x)\frac{dy}{dx} + R(x)\,P(x)\,y = R(x)\,g(x) \qquad (9.1)$$

$$\frac{d}{dx}\left(R(x)\,y\right) = R(x)\,g(x) \qquad (9.2)$$

$$\Rightarrow y = \frac{1}{R(x)}\left(\int R(x)\,g(x)\,dx + c\right)$$

where c is an arbitrary constant.

Question 9.3 *Show that the LHS of equation* (9.2) *is equivalent to the LHS of equation* (9.1).

Example 9.7 *Solve the first-order linear differential equation*

$$x\frac{dy}{dx} + 4y = x^3 - x, \quad \text{for } x > 0.$$

Solution The equation is divided through by x to give

$$\frac{\mathrm{d}y}{\mathrm{d}x} + \frac{4}{x}y = x^2 - 1.$$

1. Determine the integrating factor

Multiply both sides of the differential equation by the integrating factor (x^4) and integrate.

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(x^4 y\right) = x^4\left(x^2 - 1\right)$$

Question 9.4 *Check that* $y = \frac{x^3}{7} - \frac{x}{5} + \frac{c}{x^4}$ *is the solution for example 9.7.*

$\left(\frac{dy}{dx} + \frac{4}{x}y = x^2 - x. \right)$

9.5 Revision of key ideas

9.6 Concept map

Draw a concept map for this chapter relating the aims/key ideas of the chapter. If you are unfamiliar with the idea of a concept map see appendix A.