School of Mathematics & Applied Statistics

# MATH111: Mathematics Applied Mathematical Modelling 1

# Assignment Week 6 Solutions

# Spring 2007

1. Consider the population model

$$N_{t+1} = rN_t\left(1+\alpha N_t\right)^{-\beta},$$

where $N_{t+1}$ and $N_t$ are the populations in successive generations, $r$ is the finite net rate of increase and $\alpha$ and $\beta$ are constants defining the density dependent feedback term [1].

(a) By defining $x_t = \alpha N_t$ obtain the scaled equation

$$x_{t+1} = rx_t\left(1+x_t\right)^{-\beta},$$

showing all necessary working. What is the importance of the scaled equation?

(b) Show that the, non-negative, fixed points of the scaled equation are given by

$$x_1^* = 0,$$
$$x_2^* = (r)^{\frac{1}{\beta}} - 1.$$

What condition must be imposed upon the value of $r$ for the steady-state solution $x_2^*$ to be non-negative?

(c) Determine the stability of the trivial fixed-point solution as a function of $r$.

(d) Assuming that $\beta = 2$ determine the stability of the non-trivial fixed-point solution as a function of $r$.

(e) For the case $\beta = 2$ draw a steady-state diagram for the non-negative fixed points as a function of $r$.

[1] M.P. Hassell, J.H. Lawton and R.M. May. (1976). Patterns of Dynamical Behaviour in Single-Species Populations. *Journal of Animal Ecology* , **45**(2): 471–86.

**Solution**

(a)

$$x_t = \alpha N_t \qquad \Rightarrow N_t = \frac{x_t}{\alpha}$$
$$N_{t+1} = \frac{x_{t+1}}{\alpha}$$
$$\frac{x_{t+1}}{\alpha} = r\frac{x_t}{\alpha}\left(1+\frac{\alpha x_t}{\alpha}\right)^{-\beta},$$
$$x_{t+1} = rx_t\left(1+x_t\right)^{-\beta}.$$

The scaled equation reveals that the parameter $\alpha$ is not important as it neither determines the fixed-points nor their stability.

(b) The fixed points are found by solving the equation

$$x^* = rx^*\left(1+x^*\right)^{-\beta}$$

I drop the superscript notation from now on

$$0 = rx\left(1+x\right)^{-\beta} - x,$$
$$0 = x\left[r\left(1+x\right)^{-\beta} - 1\right].$$

Thus one fixed point is $x_1^* = 0$. The other fixed point is given by the solution of

$$\begin{aligned} 0 &= r\left(1+x\right)^{-\beta} - 1, \\ \Rightarrow \left(1+x\right)^{-\beta} &= r, \\ \Rightarrow x_2^* &= \left(r\right)^{\frac{1}{\beta}} - 1. \end{aligned}$$

For $x_2^* \geq 0$ we require $r \geq 1$.

(c) We have

$$\begin{aligned} f\left(x\right) &= rx\left(1+x\right)^{-\beta}, \\ f'\left(x\right) &= r\left(1+x\right)^{-\beta} - r\beta x\left(1+x\right)^{-\beta-1}, \\ f'\left(0\right) &= r. \end{aligned}$$

Thus the trivial fixed point, $x_1^* = 0$, is *stable* when $0 < r < 1$ and *unstable* when $r > 1$. (Note that $r$ can not be negative as it is 'the finite net rate of increase').

(d) We have

$$\begin{aligned} \left(1+x_2^*\right) &= \left(r\right)^{\frac{1}{\beta}}, \\ f'\left(x\right) &= r\left(1+x\right)^{-\beta} - r\beta x\left(1+x\right)^{-\beta-1}, \\ f'\left(x_2^*\right) &= r\left(r^{\frac{1}{\beta}}\right)^{-\beta} - r\beta x\left(r^{\frac{1}{\beta}}\right)^{-\beta-1}, \\ &= 1 - \beta x r^{-\frac{1}{\beta}}, \\ &= 1 - \beta\left(r^{\frac{1}{\beta}} - 1\right)r^{-\frac{1}{\beta}}, \\ &= 1 - \beta\left(1 - r^{-\frac{1}{\beta}}\right). \end{aligned}$$

When $\beta = 2$ this becomes

$$\lambda = f'\left(x_2^*\right) = 1 - 2\left(1 - r^{-\frac{1}{2}}\right).$$

For $r \geq 1$ we have

$$\frac{\mathrm{d}\lambda}{\mathrm{d}r} = -r^{-3/2} < 0.$$

Thus the eigenvalue decreases as $r$ increases. The minimum value of the eigenvalue occurs in the limit that $r \to \infty$ and is given by

$$\lim_{r\to\infty} \lambda = -1.$$

Thus we conclude that the fixed point $x_2^*$ is *stable* on the interval $1 < r < \infty$.

(e) See figure 1. Note that I have not plotted negative values for $x_2^*$ as these are not physically meaningful.

You should use maple for the following questions where-ever possible!

2. (a) Given the graph $y = f\left(x\right)$ explain how to obtain the graph $y = f\left(x\right) - h$, with $h > 0$.

   (b) Consider the scaled Ricker model

$$x_{n+1} = x_n \exp\left[r\left(1 - x_n\right)\right].$$

   In order to draw a cobwebbing diagram for this population model a figure containing the straight line $y = x$ and the curve $y = x\exp\left[r\left(1-x\right)\right]$ is required.

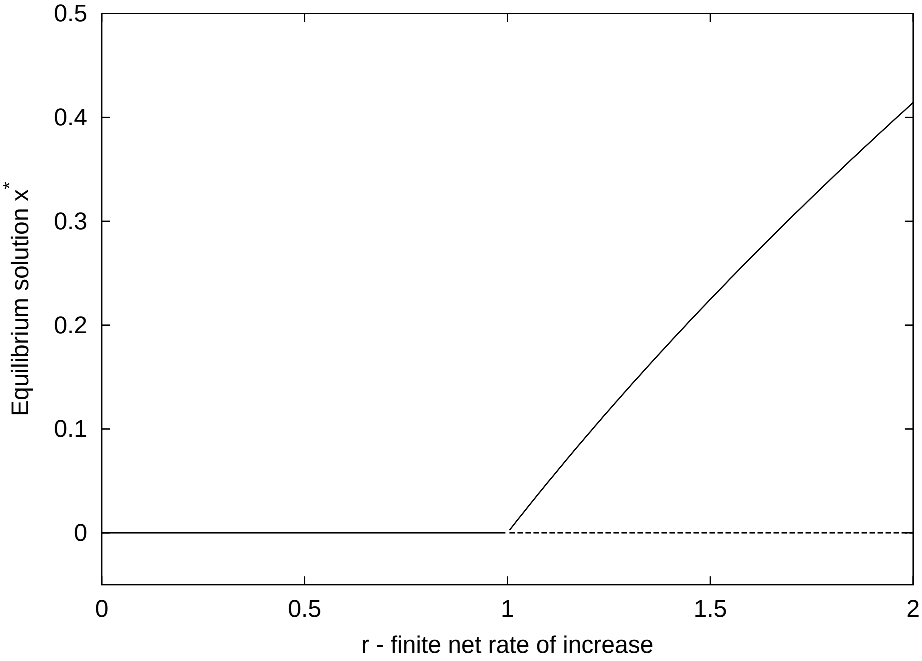



Figure 1: Steady-state diagram for the Hassell model when $\beta = 2$.

(i) Why is the straight line $y = x$ required in a cobwebbing diagram?

(ii) Write appropriate maple code to plot the straight line $y = x$ and the curve $y = x \exp[r(1-x)]$ on the same figure with $0 \le x \le 3$ and $r = 1.8$. Choose an appropriate scale for the $y$-axis.

(iii) Mark the fixed-points of the scaled Ricker model on your output. You may do this by hand.

(c) Consider the scaled Ricker model subject to fixed harvesting with $r = 1.8$

$$x_{n+1} = x_n \exp[1.8(1-x_n)] - h.$$

By providing appropriate graphs show that

(i) Harvesting is sustainable when $h = 0.25$.

(ii) Harvesting is unsustainable when $h = 1$.

(d) Estimate the critical value of the fixed harvesting parameter, $h_{\text{cr}}$, to two decimal places such that if $h = h_{\text{cr}} + 0.01$ harvesting is unsustainable whilst if $h = h_{\text{cr}} - 0.01$ harvesting is sustainable.

HINT. Use maple to plot appropriate graphs. In your answer you must make it clear how you used maple to answer this question.

**Solution**

(a) To obtain the graph $y = f(x) - h$, $h > 0$, translate the graph $y = f(x)$ down by $h$ units.

(b) (i) When a function is iterated using a cobwebbing diagram a vertical line is drawn from the initial value of $x$, $x_0$, to the curve $y = f(x)$. This gives the value $y = f(x_0) = x_1$. A horizontal line is then drawn to the line $y = x$ to obtain the value $x = y = x_1$.

(ii) See maple code at the end of this question. Note the command `y=1..1.4` to restrict the y-axis. A cobwebbing diagram should show as much of the *function* as possible.

(iii) The fixed points are where the lines $y = x$ and $y = f(x)$ intersect. They are at the points $(x, y) = (0, 0)$ and $(x, y) = (1, 1)$.

(c) (i) Harvesting is sustainable because the lines $y = x$ and $y = f(x) - 0.25$ intersect, i.e. there are fixed points. Note that I changed the $y$-scale to make this clear.

(ii) Harvesting is unsustainable because the lines $y = x$ and $y = f(x) - 1.00$ do *not* intersect, i.e. there are no fixed points. Note that I changed the $x$-scale and the $y$-scale to make this clear.

(d) The critical value of $h$ is $h_{\text{cr}} = 0.78$. Note that I changed the $x$-scale and the $y$-scale to make it clear that when $h_{\text{cr}} = 0.77$ there are two fixed points whereas when $h_{\text{cr}} = 0.77$ there are no fixed points

```
# week6harvest.maple
# 07.09.07
#
f := x;
g := x*exp(1.8*(1-x));

lprint("Graph for question (bii)");
plot({f,g},x=0..3,y=0..1.4);

lprint("Graph for question (ci)");
plot({f,g-0.25},x=0..2,y=0..1.2);

lprint("Graph for question (cii)");
plot({f,g-1.00},x=0..1.2,y=0..0.4);

lprint("Graph for question (d)");
plot({f,g-0.79,g-0.77},x=0.3..0.48,y=0.25..0.45);
```

3. Plot the following curves given in parametric form [2].

   (a) $(x, y) = \left(\frac{2}{3}\cos(t) + \frac{5}{9}\cos(2t), \frac{2}{3}\sin(t) - \frac{5}{9}\sin(2t)\right)$.
   This curve is an example of an hypocycloid. What is an hypocycloid?

   (b) $(x, y) = (\cos(3t), \sin(2t))$.
   This curve is an example of the Lissajous curve. What is the Lissajous curve?

   [2]. L. Berman. (2006). Folding Beauties. *The College Mathematics Journal* **37**(3), 176–186.

   See maple code at the end of this question. The graphs are shown in figure 2

   (a) "A hypocycloid is a special plane curve that is generated by the trace of a fixed point on a small circle that rolls within a larger circle." (wikipedia). There is a nice animation on the wikipedia entry showing the smaller circle rotating within the larger circle.

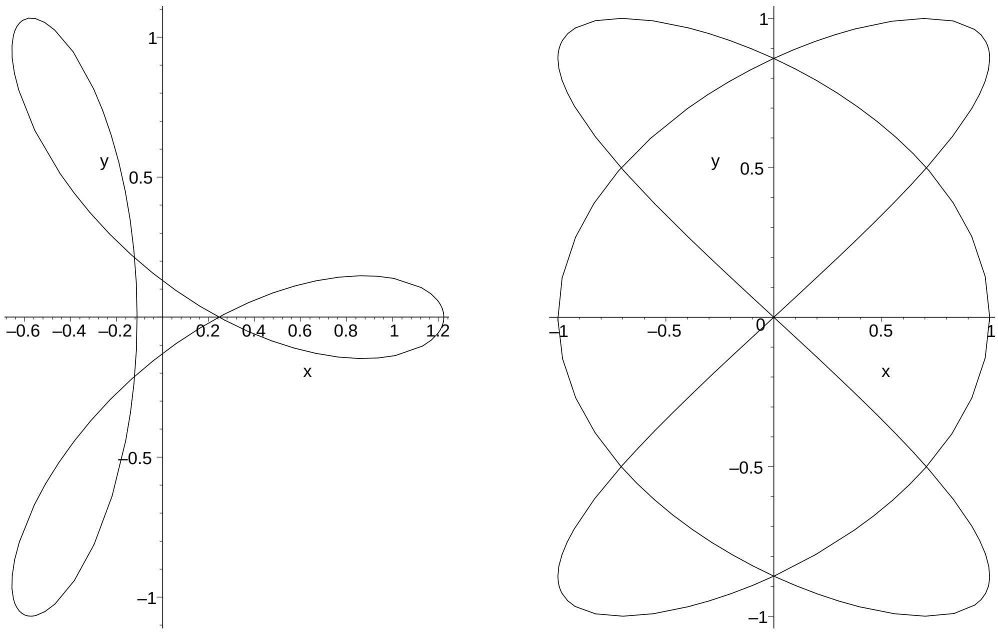
   (b) "A Lissajous curve is a curve that describes complex harmonic motion. This family of curves was investigated by Nathaniel Bowditch in 1815, and later in more detail by Jules Antoine Lissajous in 1857. The logo of the Australian Broadcasting Company is an example of a Lissajous figure." (wikipedia)

```
# week6parametric.maple
# 07.09.07
#
lprint("Question (a)");
x1 := (2/3)*cos(t) +(5/9)*cos(2*t);
y1 := (2/3)*sin(t)-(5/9)*sin(2*t);

plot([x1,y1,t=0..2*Pi],labels=["x","y"],color=black);

lprint("Question (a)");
x2 := cos(3*t);
y2 := sin(2*t);

plot([x2,y2,t=0..2*Pi],labels=["x","y"],color=black);
```




(a) A hypocycloid

(b) An example of a Lissajous curve

Figure 2: Figures for question 3.