School of Mathematics & Applied Statistics

# MATH111: Mathematics Applied Mathematical Modelling 1

# Assignment Week 10 Solutions

# Spring 2007

1. The population of fish is modelled by the differential equation

$$\frac{\mathrm{d}x}{\mathrm{d}t} = f(x),$$

where the function $f(x)$ is given by

$$f(x) = rx\left(1 - \frac{x}{K}\right)\left(\frac{x}{K_0} - 1\right),$$

where $r > 0$ and $0 < K_0 < K$.

(a) Sketch the growth curve $f(x)$ as a function of $x$.

(b) Using your sketch determine the stability of the steady-state solutions $x = 0$, $x = K_0$ and $x = K$, carefully explaining your reasoning.

(c) Calculate the stability of the steady-state solutions $x = 0$, $x = K_0$ and $x = K$ by finding their eigenvalues.

(d) How does the long-term evolution of the differential equation depend upon the choice of the initial condition $x_0$?

(e) A disease spreads through the population reducing the population density to a value $a$. What happens to the population? Justify your answer.

**Solution**

(a) See figure 1.

(b) Consider the steady-state $x_2 = K_0$. If the value of $x$ is decreased by a small amount below $x_2$ then the derivative is negative and the value of $x$ decreases away from $x_2$. If the value of $x$ is increased by a small amount above $x_2$ then the derivative is positive and the value of $x$ increase away from $x_2$. In neither case does the solution approach $x_2$. Thus the steady-state solution $x_2$ is unstable.

Consider the steady-state $x_3 = K$. If the value of $x$ is decreased by a small amount below $x_3$ then the derivative is positive and the value of $x$ increases towards $x_3$. If the value of $x$ is increased by a small amount above $x_3$ then the derivative is negative and $x$ decreases towards $x_3$. Thus the steady-state solution $x_3$ is stable. A similar argument shows that the steady-state solution $x_1 = 0$ is stable.

(c)

$$\begin{aligned}
f(x) &= r\left(x - \frac{x^2}{K}\right)\left(\frac{x}{K_0} - 1\right), \\
f'(x) &= r\left[\left(\frac{x}{K_0} - 1\right)\left(1 - \frac{2x}{K}\right) + \frac{x}{K_0}\left(1 - \frac{x}{K}\right)\right], \\
f'(0) &= -r < 0, \\
f'(K_0) &= r\left(1 - \frac{K_0}{K}\right) > 0 \quad \text{as} \quad K_0 < K \Rightarrow 1 - \frac{K_0}{K} > 0, \\
f'(K) &= r\left(1 - \frac{K}{K_0}\right) < 0 \quad \text{as} \quad K_0 < K \Rightarrow 1 - \frac{K}{K_0} < 0.
\end{aligned}$$

Thus the steady-state solution $x = K_0$ is unstable whereas the steady-state solutions $x = 0$ and $x = K$ are stable.

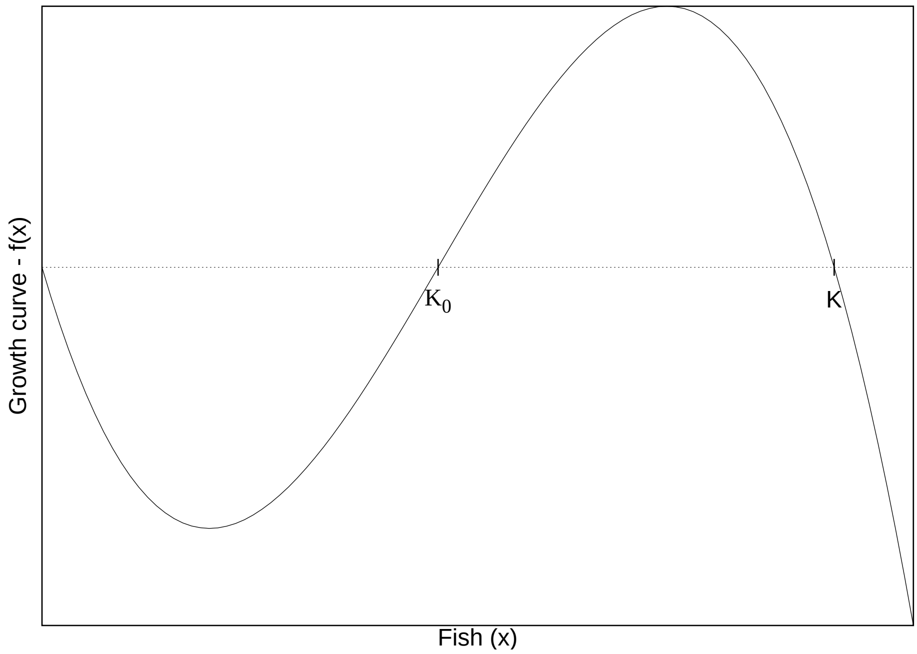


Figure 1: Variation of the growth curve ($f(x)$) with the number of fish ($x$).

(d) If $x_0 = 0$ then $x(t) = 0 \,\forall\, t$.
If $0 < x_0 < K_0$ then $x(t)$ is a decreasing function of $t$ with limit $\lim_{t\to\infty} x(t) = 0$.
If $x_0 = K_0$ then $x(t) = K_0 \,\forall\, t$.
If $K_0 < x_0 < K$ then $x(t)$ is an increasing function of $t$ with limit $\lim_{t\to\infty} x(t) = K$.
If $x_0 = K$ then $x(t) = K \,\forall\, t$.
If $K < x_0$ then $x(t)$ is a decreasing function of $t$ with limit $\lim_{t\to\infty} x(t) = K$.

(e) If $0 \le a < K_0$ then the population becomes extinct.
If $a = K_0$ then $x(t) = K_0 \,\forall\, t$.
If $K_0 < a < K$ then the population returns to the steady-state solution $X = K$.
(Justification for these statements follows from the answer to the previous question).

2. The population density of the spruce budworm in a forest is given by the differential equation

$$\frac{\mathrm{d}x}{\mathrm{d}t} = rx\left(1 - \frac{x}{q}\right) - \frac{x^2}{1+x^2}.$$

Determine the stability of the trivial steady-state solution $x = 0$.

**Solution**

$$f(x) = r\left(x - \frac{x^2}{q}\right) - \frac{x^2}{1+x^2},$$
$$\frac{\mathrm{d}f(x)}{\mathrm{d}x} = r\left(1 - \frac{2x}{q}\right) - \frac{2x}{\left(1+x^2\right)^2},$$
$$\left.\frac{\mathrm{d}f(x)}{\mathrm{d}x}\right|_{x=0} = r > 0.$$

The trivial steady-state solution is unstable.

3. The logistic equation with constant effort harvesting is given by

$$\frac{\mathrm{d}x}{\mathrm{d}t} = rx\left(1 - \frac{x}{K}\right) - Ex, \quad x(0) = K,$$

where $x$ is the population density of an animal in an environment, $t$ time, $r > 0$ is the intrinsic growth rate, $K > 0$ is the carrying capacity of the environment and $E \ge 0$ is the effort expanded in harvesting.

(a) Determine the steady-state solutions of the logistic equation with constant effort harvesting.

(b) Suppose that the value for the intrinsic growth rate is $r = 3$.

(i) Determine the stability of the steady-state solutions as a function of the effort $E$.

(ii) Draw a steady-state diagram for the logistic equation with constant effort harvesting for the case $r = 2$ showing how the steady-state solutions of the model vary as a function of the effort expanded in harvesting. Indicate stable and unstable steady-state solutions using solid and dashed lines respectively.

(iii) For what value of the parameter $E$ does a bifurcation occur? What kind of bifurcation is it?

**Solution**

(a)

$$\begin{aligned} f(x) &= rx\left(1 - \frac{x}{K}\right) - Ex, \\ &= x\left[(r - E) - \frac{rx}{K}\right]. \end{aligned}$$

The steady-state solutions are

$$\begin{aligned} x_1 &= 0, \\ x_2 &= \frac{(r - E)}{r} \cdot K. \end{aligned}$$

(b) (i)

$$\begin{aligned} f(x) &= rx\left(1 - \frac{x}{K}\right) - Ex, \\ &= r\left(x - \frac{x^2}{K}\right) - Ex, \\ f'(x) &= r\left(1 - \frac{2x}{K}\right) - E, \\ f'(x_1) &= r - E, \\ f'(x_2) &= E - r. \end{aligned}$$

The steady-state $x_1$ is stable when $r - E < 0$ and unstable when $r - E > 0$.
The steady-state $x_2$ is stable when $E - r < 0$ and unstable when $E - r > 0$.
We are told that $r = 3$. The steady-state $x_1$ is stable when $3 < E$ and unstable when $3 > E$.
The steady-state $x_2$ is stable when $E < 3$ and unstable when $E > 3$.

(ii) We are told that $r = 2$. The steady-state $x_1$ is stable when $2 < E$ and unstable when $2 > E$.
The steady-state $x_2$ is stable when $E < 2$ and unstable when $E > 2$.
The steady-state diagram is shown in figure 2.

(iii) A transcritical parameter value occurs at the parameter value $E = 2$.

4. I think that it is important that you develop an appreciation for some of the applications of mathematics in the 'real world'. The following question is designed to help you do this. It is based on Cipra (2007). You only need to write a couple of sentences to answer each of the following questions.

B.A. Cipra. 2007. Geosciences conference tackles global issues. *SIAM News*, **40**(5), pages 1, 8& 9. `www.siam.org/pdf/news/1132.pdf`

If you would like to do additional reading a good place to start is the conference web page: `www.siam.org/meetings/gs07/`.

(a) Explain how mathematical modelling can be used to enhance oil recovery. ('Simple, Flexible Models of Subsurface Flow').

(b) What is 'earthquake inversion'? Why is it important? What type of mathematical problem is it? ('Earthquake Inversion — Algorithmic Challenges')

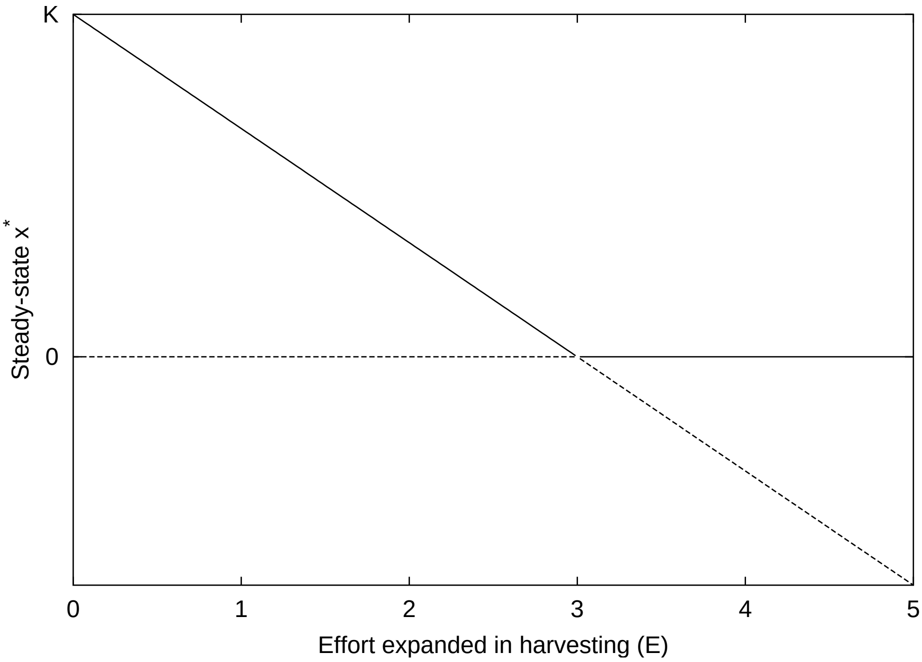


Figure 2: Steady-state diagram for the logistic differential equation with constant effort harvesting

(c) Explain one application of mathematical modelling in investigating the feasibility of carbon dioxide sequestration. ('The Modeling Problem of a Lifetime')

**Solution** No solution provided for this question.

In the the final exam you may be asked a question about Maple.

Your answer should include all maple code that you used to obtain the answer.

5. Thermal and solutal dispersion in a circular tube with diffusion into the wall is characterised by a dispersion coefficient ($\Lambda$) which is a function of the void fraction of the bed ($\epsilon_f$) and the ratio of fluid thermal diffusivity to diffusivity in the wall ($\mu$).

   Balakotaiah and Chang (2003) obtained the following formula for the dispersion coefficient

$$\begin{aligned}\Lambda &= \frac{1}{48} g_1(\epsilon_f) + \frac{\mu}{8} g_2(\epsilon_f),\\ g_1(\epsilon_f) &= \epsilon_f\left(6\epsilon_f^2 - 16\epsilon_f + 11\right),\\ g_2(\epsilon_f) &= \epsilon_f\left[4\epsilon_f - \epsilon_f^2 - 3 - 2\ln(\epsilon_f)\right].\end{aligned}$$

   (a) What is the maximum value of the function $g_1$ and for what value of the void fraction ($\epsilon_f$) does it occur?

   (b) What is the maximum value of the function $g_2$ and for what value of the void fraction ($\epsilon_f$) does it occur?

   (c) What is the maximum value of the dispersion coefficient ($\Lambda$) and the corresponding value for the void fraction ($\epsilon_f$) when the ratio of fluid thermal diffusivity to diffusivity in the wall ($\mu$) is equal to ten?

   (d) Plot the dispersion coefficient ($\Lambda$) as a function of the void fraction ($\epsilon_f$) when the ratio of fluid thermal diffusivity to diffusivity in the wall ($\mu$) is equal to 1.

   V. Balakotaiah & Chang, H-C. (2003). Hyperbolic homogenized models for thermal and solutal dispersion. *SIAM Journal on Applied Mathematics*, **63**(4), 1231–1258.

   **Solution** Maple code is at the end of the question.

(a) The maximum value for the function $g_1$ is 2.259 and it occurs when the void fraction takes the value $\epsilon_f = 0.466$.

(b) The maximum value for the function $g_1$ is 0.206 and it occurs when the void fraction takes the value $\epsilon_f = 0.139$.

(c) The maximum value for the dispersion coefficient is $\Lambda = 0.285$ and it occurs when the void fraction takes the value $\epsilon_f = 0.156$.

(d) See figure 3.

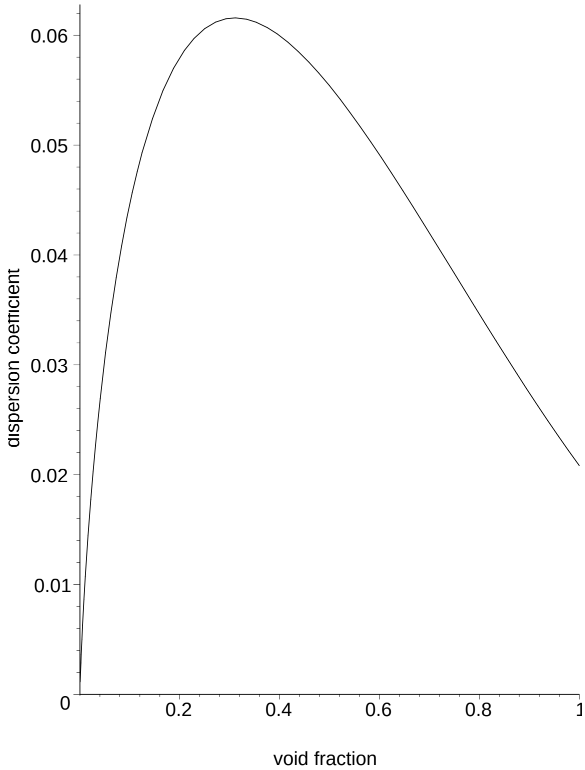


Figure 3: Variation of the dispersion coefficient ($\Lambda$) with the void fraction ($\epsilon_f$) when the ratio of fluid thermal diffusivity to diffusivity in the wall ($\mu$) is $\mu = 10$.

```
#week10-2007.maple
# 12.10.07

Lambda := g1/48.0 +mu*g2/8.0;
g1     := epsilon*(6.0*epsilon^2 -16.0*epsilon+11.0);
g2     := epsilon*(4.0*epsilon-epsilon^2-3.0-2.0*ln(epsilon));

lprint("Part (a)");

ans := solve(diff(g1,epsilon),epsilon);
# There are two solutions. We must substitute each solution into ‘g’
# to find the maximum value.
subs(epsilon=ans[1],g1);
subs(epsilon=ans[2],g1);
ans := 'ans': # reassign variable after using it.

lprint("Part (b)");
# In this part we have to use fsolve to find the value of epsilon
# where the derivative is equal to zero. We need to use the
```

```
# second derivative test to check if the solution we have found is
# a maximum point or a minimum point.

g2e := diff(g2,epsilon):
g2ee := diff(g2e,epsilon):

ans := fsolve(g2e,epsilon);
evalf(subs(epsilon=ans,g2));
subs(epsilon=ans,g2ee);
ans := 'ans':

lprint("Part (c)");
# In this part we have to use fsolve to find the value of epsilon
# where the derivative of Lambda is equal to zero. We need to use the
# second derivative test to check if the solution we have found is
# a maximum point or a minimum point.


mu  := 10;
Le  := diff(Lambda,epsilon):
Lee := diff(Le,epsilon):

ans := fsolve(Le,epsilon);
evalf(subs(epsilon=ans,Lambda));
subs(epsilon=ans,Lee);
ans := 'ans':

lprint("Part (d)");
mu := 1;
plot(Lambda,epsilon=0..1,labels=["void fraction","dispersion coefficient"],\
     labeldirections=[horizontal,vertical],color=black);
```