School of Mathematics & Applied Statistics

MATH111: Mathematics Applied Mathematical Modelling 1

Assignment Week 6 Solutions

Spring 2006

1. Consider the logistic equation with fixed harvesting

$$x_{n+1} = rx_n(1 - x_n) - h, \quad n = 0, 1, 2 \dots$$

where $1 < r < 4$ and $0 \le h \le 1$.

(a) The stable fixed point of the harvesting model (should it exist) is given by

$$x^* = \frac{-(1-r) + \sqrt{(1-r)^2 - 4rh}}{2r}$$

Find the fixed point (to four decimal places), and the associated eigenvalue, when

(i) $r = 1.5$ and $h = 0.015$.
(ii) $r = 1.6$ and $h = 0.05625$.
(iii) $r = 2$ and $h = 0.045$.

Solution Let x^* be a fixed point of the map

$$x_{n+1} = f(x_n),$$

then its eigenvalue is given by

$$\lambda(x^*) = f'(x^*).$$

For the map

$$x_{n+1} = rx_n(1 - x_n) - h,$$

we have

$$\lambda(x^*) = r(1 - 2x^*).$$

It is straightforward to show that the stable fixed point and the associated eigenvalue are given by

(i) $x^* = 0.3$, $\lambda = 0.6$.
(ii) $x^* = 0.1875$, $\lambda = 1.0$.
(iii) $x^* = 0.45$, $\lambda = 0.2$.

(b) Gollum Fresh Fish (motto 'fish fresh from the sea, three times a day') has the choice to send its fishing fleet to one of three fisheries. The value for h is regulated by Mordor Moguls. The long-term yearly profit ($\mathcal{P}$) for fishing in a fishery is

$$\mathcal{P} = ax^* - b$$

where a and b are parameters that depend upon the fishery and x^* is the steady fixed point of the harvesting model for the specified values of h and r. The numbers associated with each fishery are

Fishery one: $r = 1.5$, $h = 0.015$, $a = 3$, $b = 0.6$.
Fishery two: $r = 1.6$, $h = 0.05625$, $a = 4$, $b = 0.45$.
Fishery three: $r = 2$, $h = 0.045$, $a = 1$, $b = 0.2$.

Which fishery should Gollum Fresh Fish use and why?

Solution Using our answer to part (a) it is easy to show that the profit in the three fisheries are

Fishery one : 0.3.

Fishery two : 0.3.

Fishery three : 0.25.

Gollum Fresh Fish should use fishery one because the eigenvalue is 0.6. At fishery two the eigenvalue is 1.0 — fishing is on the edge of stability.

2. Show that the unscaled logistic model with proportional harvesting

$$N_{n+1} = N_n \left(r - \frac{N_n}{K} \right) - pN_n \quad 0 \leq p$$

can be written in its standard form

$$x_{n+1} = rx_n \left(1 - x_n\right) - px_n$$

by introducing a new scaling $x_n = \frac{N_n}{rK}$.

Solution

$$\begin{aligned} N_n &= rKx_n, \\ N_{n+1} &= rKx_{n+1}, \\ rKx_{n+1} &= rKx_n \left(r - \frac{rKx_n}{K} \right) - prKx_n, \\ x_{n+1} &= x_n \left(r - rx_n \right) - px_n, \\ &= rx_n \left(1 - x_n\right) - px_n. \end{aligned}$$

3. (a) Find the nonnegative equilibria of a population governed by

$$x_{n+1} = \frac{3x_n^2}{x_n^2 + 2}$$

and check for stability.

Solution The fixed points of the difference equation

$$x_{n+1} = f\left(x_n\right)$$

are found by solving the equation

$$x^* = f\left(x^*\right).$$

For notational convenience I will drop the superscript * notation in our example

$$\begin{aligned} x &= \frac{3x^2}{x^2 + 2}, \\ x\left(x^2 + 2\right) &= 3x^2, \\ x^3 - 3x^2 + 2x &= 0, \\ x\left(x^2 - 3x + 2\right) &= 0, \\ x\left(x - 1\right)\left(x - 2\right) &= 0. \end{aligned}$$

Therefore the non-negative fixed points are $x_1^* = 0$, $x_2^* = 1$, $x_3^* = 2$.

Suppose that x^* is a fixed point of the difference equation

$$x_{n+1} = f\left(x_n\right).$$

Then the eigenvalue of the fixed point is given by

$$\lambda\left(x^*\right) = f'\left(x^*\right).$$

In the current example we have

$$f\left(x\right) = \frac{3x^2}{x^2+2}.$$
$$\Rightarrow f'\left(x\right) = \frac{12x}{\left(x^2+2\right)^2}.$$

Thus we have:

$$\lambda\left(x_1^*\right) = 0 \quad \text{and the fixed point } x_1^* = 0 \text{ is stable;}$$
$$\lambda\left(x_2^*\right) = \frac{12}{9} \quad \text{and the fixed point } x_2^* = 1 \text{ is unstable;}$$
$$\lambda\left(x_3^*\right) = \frac{24}{25} \quad \text{and the fixed point } x_3^* = 2 \text{ is stable.}$$

(b) Suppose a fraction a is removed from the population in each generation, so that the model becomes

$$x_{n+1} = \frac{3x_n^2}{x_n^2+2} - ax_n.$$

For what values of a is there a stable equilibrium only at $x = 0$?
Solution The fixed points of the difference equation

$$x_{n+1} = f\left(x_n\right)$$

are found by solving the equation

$$x^* = f\left(x^*\right).$$

For notational convenience I will drop the superscript * notation in our example

$$x = \frac{3x^2}{x^2+2} - ax,$$
$$x\left(x^2+2\right) = 3x^2 - ax\left(x^2+2\right),$$
$$\left(a+1\right)x^3 - 3x^2 + 2\left(1+a\right)x = 0,$$
$$x\left[\left(1+a\right)x^2 - 3x + 2\left(1+a\right)\right] = 0.$$

Thus the fixed points are

$$x_1^* = 0,$$
$$x_\pm^* = \frac{3 \pm \sqrt{9 - 8\left(1+a\right)^2}}{2\left(1+a\right)}.$$

Thus there will be only one fixed point when

$$9 - 8\left(1+a\right)^2 < 0, 9 < 8\left(1+a\right)^2,$$
$$\sqrt{\frac{9}{8}} < 1 + a,$$
$$\frac{3}{2\sqrt{2}} - 1 < a,$$
$$\Rightarrow a > \frac{3 - 2\sqrt{2}}{2\sqrt{2}},$$
$$a > \frac{3\sqrt{2} - 4}{4}.$$

The stability of the fixed-point $x_1^* = 0$ is given by

$$\lambda(x) = \frac{12x}{\left(x^2+2\right)^2} - a,$$
$$\lambda(x_1^* = 0) = -a.$$

Thus the fixed points $x_1^* = 0$ is stable when

$$0 < a < 1.$$

Thus the range of values of a over which there is a stable equilibrium only at $x = 0$ is

$$\frac{3\sqrt{2}-4}{4} < a < 1.$$

In the mid-session test and/or the final exam you may be asked a question about Maple.

4. The following example is taken from the lecture notes.

   The interest on an investment of \$5 000 at the MAS Bank earns 6.5% interest rate compounded monthly. The teller at the bank explains that at the end of every month, the new principal will be worked out using the equation
   $$P_{n+1} = \left(1 + \frac{0.065}{12}\right) P_n\,, \qquad P_0 = 5\,000, \quad n = 0, 1, 2, \ldots \tag{1}$$
   How much money is in the bank account at the end of the first month?

   The answer to this question is that $P_1 = \$5,027.08$. At the Wollongong campus a student rounded their answer down to \$5,027.

   Before answering this question you are advised to carefully consider **question 6** from the week two maple worksheet.

   (a) A student invests \$5,000 at the MAS Bank at 6.5% compounded monthly. How much money do they have after twenty years? (You do *not* need maple for this part of the question!)

   Solution We have a first-order linear difference equation of the form
   $$x_n a x_{n-1}.$$
   The solution of this equation is
   $$x_n = a^n x_0.$$
   Thus the money after twenty years is given by
   $$P_{240} = \left(1 + \frac{0.065}{12}\right)^{240} 5000,$$
   $$= \$18,282,23.$$
   Note that $n = 240$ because interest is compounded monthly for twenty years.

   (b) Suppose that at the end of each month MAS Bank rounds your investment down to the nearest dollar. How much money have you lost to the bank after twenty years?

   Solution If, at the end of every month. MAS Bank rounds your investment down to the nearest dollar then after twenty years you have \$18039 in your account. Thus you have lost \$243.23.

   Here's my maple code for this question.

```
# bank.maple (31.08.06)
# A simple maple program to iterate the first-oder difference
# equation
# x_{n+1} = (1+growth)x_{n};

finalyear := 240;    # The final value of 'n'.
growth  := 0.065/12; # the monthly interest rate.
year  := n->n;      # define the 'time' variable.

f := x -> (1+growth)*x; # x_{n+1} = f(x_{n})

# Instead of using the variable x_{n} we will use the variable
# money_{n}
money := proc(n)          # define the values of x_{n} recursively.
        option remember;  # Note that using the option remember causes the
        floor(f(money(n-1))) # previous values y(n-1) to be retained so that
        end:              # subsequent values may be based on them.

money(0) := 5000;         # the initial amount of money invested.

# calculate the ordered pairs (money{n},time_{n}) for n=0..finalyear

solution := [seq([year(n),money(n)],n=0..finalyear)];
```