School of Mathematics & Applied Statistics

MATH111: Mathematics Applied Mathematical Modelling 1

Assignment Week 4 Solutions

Spring 2006

1. Suppose that at time $t = 0$ you borrow an amount y_0 from a bank and thereafter you borrow a an amount B at regular intervals. You pay compound interest on your total borrowings.

   (a) Write a word equation to describe this scenario.

   (b) Convert your word equation to a difference equation;

   (c) Solve your difference equation to find the amount of money you owe at time n.

   (d) Suppose that at the start of January 1994 Belinda and Peter borrow \$2,000 and at the start of each subsequent month they borrow an additional \$500. At the start of January 2004 they have borrowed a total of \$62,000. What is their accumulated debt if they are being charged 8.5% interest compounded monthly?

   (e) The equation that you have derived is the same as the equation for annuities. Explain why this makes sense.

Solution.

(a)

$$\left\{ \begin{array}{c} \text{Change in} \\ \text{money owed} \end{array} \right\} = \left\{ \begin{array}{c} \text{interest on money} \\ \text{owed last interval} \end{array} \right\} + \left\{ \begin{array}{c} \text{money borrowed} \\ \text{this interval} \end{array} \right\}$$

(b) Let y_n be the amount of money owed at the end of the nth interval, α be the fraction of the year between borrowings, $p/100$ be the yearly compound interest rate and B be the amount of money borrowed every interval. Then

$$y_n = \left(1 + \frac{\alpha p}{100}\right) y_{n-1} + B.$$

(c) The solution of the general difference equation

$$x_n = a x_{n-1} + b\,(n)$$

is given by

$$x_n x_0 a^n + \sum_{p=1}^{n} a^{n-p} b\,(p)\,.$$

Applying this solution to the equation we derived in part (b) we obtain

$$\begin{aligned} y_n &= y_0 \left(1 + \frac{\alpha p}{100}\right)^n + \sum_{p=1}^{n} \left(1 + \frac{\alpha p}{100}\right)^{n-p} B, \\ &= y_0 \left(1 + \frac{\alpha p}{100}\right)^n + B \sum_{p=1}^{n} \left(1 + \frac{\alpha p}{100}\right)^{n-p}, \\ &= y_0 \left(1 + \frac{\alpha p}{100}\right)^n + B \left[\frac{\left(1 + \frac{\alpha p}{100}\right)^n - 1}{\frac{\alpha p}{100}} \right], \\ &= \left(1 + \frac{\alpha p}{100}\right)^n \left(y_0 + \frac{100B}{\alpha p}\right) - \frac{100B}{\alpha p}. \end{aligned}$$

(d) We have

$$\begin{aligned} y_0 &= 2000, \\ B &= 500, \\ \alpha &= \frac{1}{12}. \\ p &= 8.5, \\ n &= 120, \end{aligned}$$

Using the formula we obtained at the end of part (c) the accumulated debt is

$$y_{120} = \$98,734.50$$

(e) Consider the scenario from the perspective of the bank. At time $t = 0$ they 'opened' an account with Peter and Belinda in which they placed $y_0 = 2000$. Subsequently they are depositing an amount \$500 at regular intervals. Their investment draws interest of 8.5% compounded monthly. This is an annuity.

2. Suppose that in January 1994 Belinda and Peter had \$1000 to invest. They invested it in an Australian equity fund and contributed \$250 per month for the next 10 years.

(a) What is their total investment after 10 years?

(b) If over the period of investment the average annual return on the fund is 10.6% how much would money would they have in their equity fund?

(c) Borrowing money to invest it called gearing. It is a popular investment strategy.

Suppose that in January 1994 Belinda and Peter borrowed \$2000 to add to their initial investment of \$1000 and that thereafter they borrowed an additional \$500 a month to add to the \$250 they investested. At the end of 10 years how much would they have in their equity fund?

(d) At the end of ten years Belinda and Peter close their equity fund. How much money do they have after they have paid off the accumulated debt on their loan? (Use your answer to question 1 (d).

(Based on an example in. "We can open your eyes to a world of investment options". Bridges. 2006.

Solution.

(a) Their total investment is

$$1000 + 12 \times 250 \times 10 = \$31,000.$$

(b) In effect the equity fund is operating as an annuity with

$$\begin{aligned} y_0 &= 1000, \\ I &= 250, \\ \alpha &= \frac{1}{12}, \\ p &= 10.6, \\ n &= 12 \times 10. \\ y_n &= \left(1 + \frac{\alpha p}{100}\right)^n \left(y_0 + \frac{100I}{\alpha p}\right) - \frac{100I}{\alpha p}, \\ &= \$55,881.50. \end{aligned}$$

(c) Repeat the previous calculation with

$$\begin{aligned} y_0 &= 3000, \\ I &= 750, \\ y_n &= \left(1 + \frac{\alpha p}{100}\right)^n \left(y_0 + \frac{100I}{\alpha p}\right) - \frac{100I}{\alpha p}, \\ &= \$167,644.50. \end{aligned}$$

(d) From our answer to question 1 (d) we know that Peter and Belinda owe the bank \$ 98,734.50. Thus their net profit is

$$\$167,644.50 - \$98,734.50 = \$68,910.$$

3. Lien borrows \$20,000 to have a MATH111 chip implanted in her head so that everything makes sense. Interest is compounded monthly at 9% p.a.

   (a) In the first year Lien makes no repayments. How much does she owe at the end of the year?

   (b) Starting in the second year Lien makes a repayment at the end of each month. If the loan is to be repaid after a further nine years what is the monthly repayment?

   Solution.

   (a) Use the compound interest equation with

$$\begin{aligned} S_0 &= 20,000, \\ \alpha &= \frac{1}{12}, \\ p &= 9, \\ n &= 12, \\ S_n &= S_0\left(1 + \frac{\alpha p}{100}\right)^n, \\ &= \$21,876.13796 \end{aligned}$$

   (b) Use the loan repayment formula with

$$\begin{aligned} D_0 &= 21,987.13796, \\ \alpha &= \frac{1}{12}, \\ p &= 9, \\ n &= 108, \\ D_{108} &= 0, \\ D_n &= \left(1 + \frac{\alpha p}{100}\right)^n \left(D_0 - \frac{100R}{\alpha p}\right) + \frac{100R}{\alpha p}, \\ \Rightarrow R &= \$296.27. \end{aligned}$$

4. You have been given \$1000 to invest for one year. You have a choice of three bank accounts.

   - 'You Beaut' bank offers you 10% p.a. compounded annually. At the end of the year you will pay \$20 in fees.
   - 'Fair Go' bank offers you 10% p.a. compounded quarterly. At the end of the year you will pay \$30 in fees.
   - 'Ocker' bank offers you 11% p.a. compounded every four months. At the end of the year you will pay \$25 in fees.

   Which bank should you put your money into (justify your answer)?

   Solution. The money in the bank at the end of the year ($\mathcal{M}$) is the total investment, determined using the compound interest formula, minus the bank fee.

You Beaut

$$\mathcal{M} = 1000\left(1 + \frac{1 \times 10}{100}\right) - 20 = 1080.$$

Fair Go

$$\mathcal{M} = 1000\left(1 + \frac{10}{400}\right)^4 - 30 \quad = 1073.81,$$

Ocker

$$\mathcal{M} = 1000\left(1 + \frac{11}{300}\right)^3 - 25 \quad = 1089.08.$$

Therefore you should put your money into Ocker as it gives the largest return.

5. Suppose that in a battle between two opposing forces each unit of army X is able to destroy b units of army Y during one time unit. Similarly each unit of army Y is able to destroy a units of army X.

   (a) Write down two **word** equations that define the problem — one for each army.

   (b) Write down, formally, the two difference equations that describe the above scenario — one difference equation for each army. Define **all** variables and explain your terms.

Solution.

(a)

$$\left\{\begin{array}{c}\text{Change in the}\\ \text{number of units of } X\end{array}\right\} = \left\{\begin{array}{c}\text{number of units}\\ \text{destroyed by } Y\end{array}\right\},$$

$$\left\{\begin{array}{c}\text{Change in the}\\ \text{number of units of } Y\end{array}\right\} = \left\{\begin{array}{c}\text{number of units}\\ \text{destroyed by } X\end{array}\right\}.$$

(b) Let X_n and Y_n be the number of units of army X and army Y that are present in time unit n. The number of units of X destroyed by army Y in the $n-1$th time interval is aX_{n-1}. The number of units of Y destroyed by army X in the $n-1$th time interval is bY_{n-1}. Thus the model is given by

$$X_n = X_{n-1} - aY_{n-1},$$
$$Y_n = Y_{n-1} - bX_{n-1}.$$

In the mid-session test and/or the final exam you may be asked a question about Maple.

6. The specific growth rate (μ) of microorganisms/enzymes utilizing an inhibitory substrate is given by (1)

$$\mu = \frac{\mu_{\max} S}{K_s + S + \beta S^2},$$

where $\mu_{\max}$ is the maximum specific growth rate in the absence of inhibition (hr^{-1}), S is the substrate concentration ($\text{g}\,\text{l}^{-1}$), K_s is the saturation constant ($\text{g}\,\text{l}^{-1}$), and β is the reciprocal of the inhibition constant ($\text{l}\,\text{g}^{-1}$). All constants are positive.

(1) J.F. Andrews. (1968). A Mathematical Model for the Continuous Culture of Microorganisms Utilizing Inhibitory Substrates. *Biotechnology and Bioengineering* **10**(6): 707–723.

Before answering this question you are advised to read sections 4.1 & 4.2 of the "Introduction to Maple" book. You are also advised to read the Maple help pages for the commands `diff`, `plot`, `solve`, and `plots[display]`. You may need to load the `plots` package into maple.

   (a) What is the maximum value of the specific growth rate and for what substrate concentration does it occur?

(b) Assume that $\mu_{\max} = 1\,\mathrm{h}\,\mathrm{t}^{-1}$ and that $K_s = 0.03\,\mathrm{g}\,\mathrm{l}^{-1}$. On one figure plot the specific growth rate as a function of the substrate concentration, over the range $0 \le S\left(\mathrm{g}\,\mathrm{l}^{-1}\right) \le 2.0$, for the following values of the inhibition constant: $\beta = 0\left(\mathrm{l}\,\mathrm{g}^{-1}\right)$, $\beta = 0.5\left(\mathrm{l}\,\mathrm{g}^{-1}\right)$ and $\beta = 2\left(\mathrm{l}\,\mathrm{g}^{-1}\right)$.
Your graph should contain appropriate labels for the axis.

Your answer to (i) & (ii) should include all maple code that you used to obtain the answer.

Solution

(a) The maximum value occurs when $S = \sqrt{K_s\beta}/\beta$ and is given by $\mu\left(S = \sqrt{K_s\beta}/\beta\right) = \frac{\mu_{\max}}{1+2\sqrt{K_s\beta}}$.

(b) See the figure.

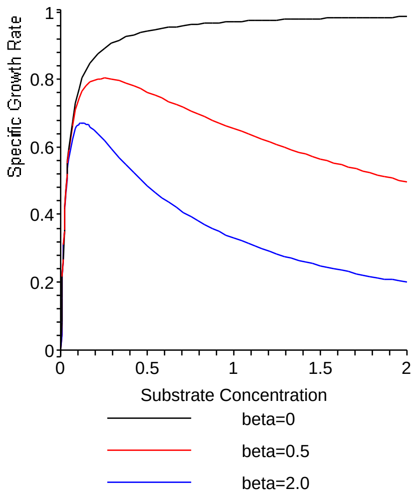

Figure 1: Figure for question (6b) showing the specific growth rate rate as a function of substrate concentration. Parameter values: $\mu_{\max} = 1\,\mathrm{hr}^{-1}$, $K_s = 0.03\mathrm{gl}^{-1}$.

Here's my maple code for this question.

```
# week4-2006.maple
# 11.08.06
#

mu := mumax*S/(K+S+beta*S^2);

muS := diff(mu,S);
solve(muS,S);
# We are only interested in the positive solution.
lprint("The maximum value of the growth rate functions occurs when");
S := sqrt(K*beta)/beta;
lprint("The maximum value of the growth rate function is given by");
simplify(mu);
# It's easy to see by inspection that the maximum value is given by
# mumax/(1+2sqrt(K*beta))

S := 'S';
mumax := 1:
K     := 0.03:

# Case One.
beta := 0:
plot1 :=plot(mu,S=0..2,colour=black,legend=("beta=0")):
```

```
#Case Two.
beta := 0.5:
plot2 := plot(mu,S=0..2,colour=red,legend=("beta=0.5")):

#Case Three.
beta := 2.0:
plot3 := plot(mu,S=0..2,colour=blue,legend=("beta=2.0")):

with(plots):
display({plot1,plot2,plot3}, labeldirections=[horizontal,vertical],\
        labels=["Substrate Concentration","Specific Growth Rate"],\
        thickness=1);
```