School of Mathematics & Applied Statistics

MATH111: Mathematics Applied Mathematical Modelling 1

Assignment Week 2 Solutions

Spring 2006

1. For the following difference equations: identify the state variable and the 'time' variable; give their order and state whether they are linear, nonlinear, autonomous or non-autonomous.

   (a) $n_{y+2} = n_{y+1}y$

   (b) $y_{n+1} = 2y_{n-1} + \sin(n)$

   Solution.

   State variable.
   In (a) the state variable is n. In (b) the state variable is y.

   Time variable.
   In (a) the 'time' variable is y. In (b) the 'time' variable is n.

   Order.
   If we denote P and Q to be the largest and smallest subscripts on the variable that occur in the difference equation, then the **order** of the difference equation is given by $P - Q$.

   In (a) the order is $(y+2) - (y+1) = 1$.
   In (b) the order is $(n+1) - (n-1) = 2$.

   Linear or non-linear?.
   A difference equation is said to be linear if each of the states $\ldots, x_{n-2}, x_{n-1}, x_n, x_{n+1}, x_{n+2}, \ldots$ appearing in the equation are present linearly. Otherwise it is termed **non-linear**.

   In (a) n_{y+2} and yn_{y+1} are both linear terms. Thus the equations is linear.

   In (b) y_{n+1} and $2y_{n-1}$ are both linear. Thus the equation is linear — note we do not examine the term $\sin(n)$ as this expression does not contain a state variable.

   Autonomous or non-autonomous?.
   If a difference equation contains the 'time' variable explicitly in the equation, then we refer to it as an non-autonomous difference equation. Otherwise it is called **autonomous**.

   In (a) the subscript is y. This appears in the equation as $n_{y+1}y$. Thus the equation is non-autonomous.

   In (b) the subscript is n. This appears in the equation as $\sin(n)$. Thus the equation is non-autonomous.

2. (a) Show that $x_n = an + b$ is a solution of the difference equation

   $$x_{n+1} - 2x_n + x_{n-1} = 0,$$

   where a and b are constants.

   (b) Find the solution of the difference equation

   $$n_{x+1} - 2n_x + n_{x-1} = 0, \qquad n(x=1) = 7,\ n(x=3) = 13.$$

   Solution

   (a) From the proposed we have

   $$\begin{aligned} x_{n+1} &= a(n+1) + b, \\ x_n &= an + b, \\ x_{n-1} &= a(n-1) + b. \end{aligned}$$

Substituting these expressions into the LHS of the difference equation we have

$$\begin{aligned} x_{n+1} - 2x_n + x_{n-1} &= a(n+1) + b - 2(an+b) + a(n-1) + b, \\ &= (an - 2an + an) + (a - a) + (b - 2b + b), \\ &= 0, \\ &= \text{RHS. Thus } x_n = an + b \text{ is a solution of the difference equation.} \end{aligned}$$

(b) From our answer to part (b) we know that the general solution is given by

$$n_x = ax + b \ .$$

From the information in the question we have

$$\begin{aligned} n_1 = a + b &= 7, \\ n + 3 = 3a + b &= 13. \end{aligned}$$

It follows that

$$a = 3 \qquad b = 4.$$

The solution of the difference equation is therefore

$$n_x = 3x + 4.$$

3. Solve the following difference equations to obtain solutions in "closed form".

(a) $y_{n+1} - \frac{1}{2}y_n = 2, y_0 = c$.

(b) (i) Evaluate the expression $\sum_{p=1}^{n} 2^{n-p} \cdot 3^p$. Hint Show that

$$\sum_{p=1}^{n} 2^{n-p} \cdot 3^p = 2^n \left[\left(\frac{3}{2}\right) + \left(\frac{3}{2}\right)^2 + \ldots + \left(\frac{3}{2}\right)^n \right].$$

(ii) Hence solve the difference equation $x_n = 2x_{n-1} + \dfrac{3^n}{3}, \quad x_0 = 0.5.$

Solution.

The solution to the equation

$$x_n = ax_{n-1} + b(n)$$

is

$$x_n = a^n x_0 + \Sigma_{p=1}^{n} b(p)\, a^{n-p}.$$

(a) $y_{n+1} - \frac{1}{2}y_n = 2$, $y_0 = c$. Put this equation in the general form by substituting $n \to n-1$. Then the equation becomes $y_n - \frac{1}{2}y_{n-1} = 2$, $y_{-1} = c$. We have

$$\begin{aligned} a &= \frac{1}{2}, \\ b(n) &= 2. \end{aligned}$$

Thus the solution is

$$\begin{aligned} y_n &= \left(\frac{1}{2}\right)^n y_0 + \sigma_{p=1}^{n} 2 \left(\frac{1}{2}\right)^{n-p}, \qquad \text{where } y_0 \text{ is the initial condition.,} \\ &= \left(\frac{1}{2}\right)^n y_0 + 2\sigma_{p=1}^{n} \left(\frac{1}{2}\right)^{n-p}, \\ &= \left(\frac{1}{2}\right)^n y_0 + 2 \cdot 1 \left[\frac{\left(\frac{1}{2}\right)^n - 1}{\frac{1}{2} - 1} \right], \end{aligned}$$

(as the term inside the summation sign is a geometric progression with $k = 1$ and $r = \frac{1}{2}$).

$$= 4 + \left(\frac{1}{2}\right)^n (y_0 - 4).$$

Now we have

$$\begin{aligned} y_0 &= \frac{1}{2} y_1 + 2, \\ &= \frac{1}{2} c + 2. \end{aligned}$$

Thus the solution is

$$y_n = 4 + \left(\frac{1}{2}\right)^n \left(\frac{1}{2} c - 2\right).$$

(b) (i)

$$\begin{aligned} \mathcal{I} &= \sum_{p=1}^{n} 2^{n-p} \cdot 3^p, \\ &= 2^{n-1} \cdot 3 + 2^{n-2} \cdot 3^2 + \ldots 2^1 \cdot 3^{n-1} + 2^0 \cdot 3^n, \\ &= 2^n \left(2^{-1} \cdot 3 + 2^{-2} \cdot 3^2 + \ldots + 2^{n-1} \cdot 3^{n-1} + 2^{-n} \cdot 3^n\right), \\ &= 2^n \left[\left(\frac{3}{2}\right) + \left(\frac{3}{2}\right)^2 + \ldots + \left(\frac{3}{2}\right)^{n-1} + \left(\frac{3}{2}\right)^n\right]. \end{aligned}$$

Note that the term inside the brackets is a geometric progression with $k = \frac{3}{2}$ and $r = \frac{3}{2}$. It's sum is $3\left[\left(\frac{3}{2}\right)^n - 1\right]$. Thus

$$\mathcal{I} = 3 \cdot 2^n \left[\left(\frac{3}{2}\right)^n - 1\right].$$

(ii)

$$\begin{aligned} a &= 2, \\ b(n) &= \frac{3^n}{3}, \\ x_0 &= 0.5. \end{aligned}$$

Thus the solution is

$$\begin{aligned} x_n &= (2)^n \cdot \frac{1}{2} + \sum p = 1^n (2)^{n-p} \frac{3^p}{3}, \\ &= 2^{n-1} + \frac{1}{3} \sum p = 1^n (2)^{n-p} 3^p, \\ &= 2^{n-1} + 2^n \left[\frac{3}{2}^n - 1\right], \\ &= 2^{n-1} + 3^n - 2^n, \\ &= 3^n - 2^{n-1}. \end{aligned}$$

4. Consider the problem of modelling the number of of chickens in Mr & Mrs Tweedy's farm. Each week the following activities occur:

- The number of chickens increases through natural growth by 10%.
- A fraction, α, of the chickens are killed by foxes.

- A constant number of chickens are converted into chicken pies.

(a) Write down a **word** equation that defines this problem.

(b) Write down, formally, the difference equation that describes the above scenario. Define **all** variables and explain your terms.

Solution

(a)

$$\left\{ \begin{array}{c} \text{Change in} \\ \text{chicken numbers} \end{array} \right\} = \left\{ \begin{array}{c} \text{net} \\ \text{growth} \end{array} \right\} - \left\{ \begin{array}{c} \text{chickens killed} \\ \text{by foxes} \end{array} \right\} - \left\{ \begin{array}{c} \text{chickens converted} \\ \text{to pies} \end{array} \right\}$$

(b) Let C_w and C_{w-1} be the number of chickens on the farm in weeks w and $w-1$ respectively.
Let g be the net growth rate. Let α be the fraction of chickens that are killed by foxes.
Let N be the number of chickens that are converted into pies.
Then

$$C_w - C_{w-1} = gC_{w-1} - \alpha C_{w-1} - N,$$
$$\Rightarrow C_w = (1 + g - \alpha)\, C_{w-1} - N.$$

The question states that $g = 0.1$. Thus

$$C_w = (1.1 - \alpha)\, C_{w-1} - N.$$

5. Municipal solid waste (MSW) may contain upto 30-40% of organic materials by mass. These organic wastes should be removed from the MSW before the MSW is delivered to a landfill site. One way to do this is to biologically oxidise the organic fraction.

The maximum specific oxidation rate (hr^{-1}) in a bioreactor is given by the formula

$$\mu_{\max} = \frac{A' \exp\left[-\frac{E_g}{RT}\right]}{1 + B \exp\left[-\frac{\Delta G_d}{RT}\right]},$$

where $A'\left(\text{hr}^{-1}\right)$ and $B\,(-)$ are constants, E_g is the activation energy of the growth process (kJ/mol), and ΔG_d is the Gibbs free energy change upon protein denaturation (kJ/mol).

In (1) the following parameter values were estimated: $A' = 4.032 \times 10^8\,\text{hr}^{-1}$, $B = 4.776 \times 10^{89}$, $E_g = 56.861\,\text{kJ/mol}$, $\Delta G_d = 537.56\,\text{kJ/mol}$, $R = 8.31431\,\text{kJ/mol/K}$.

(1) E. Liwarska-Bizukojc, M. Bizukojc and S. Ledakowicz. 2001. Kinetic model for the process of aerobic biodegradation of organic fraction of municipal solid waste. *Bioprocess and Biosystems Engineering*, **24**: 195–202.

(a) Plot the function $\mu_{\max}$ as a function of temperature over the range $275 \leq T\,(\text{K}) \leq 320$.

(b) Find the value of T (to one decimal place) that maximises the value of $\mu_{\max}$.

Solution.

(a) Here's my maple code for both parts of this question and the figure it generated.

```
# week2-2006.maple  (i) Plot the maximum specific oxidation rate
# 29.07.06              as a function of temperature.
#                   (ii) Find the value of T that maximum the oxidation rate.
#
# Define the function
#
mumax := A*exp(-Eg/(R*T))/ (1+B*exp(-Gd/(R*T)));

# Define the constants
```

```
A  := 4.032e8;
B  := 4.776e89;
Eg := 56.861e3;
Gd := 537.56e3;
R  := 8.31431;

# PART (i) plot the function

plot(mumax,T=275..320,labels=["Temperature","Maximum specific oxidation rate"],\
labeldirections=[horizontal,vertical]);

# PART (ii) differentiate the function. Then solve the equation to find T.

eqn := diff(mumax,T);
solve(eqn,T);
```

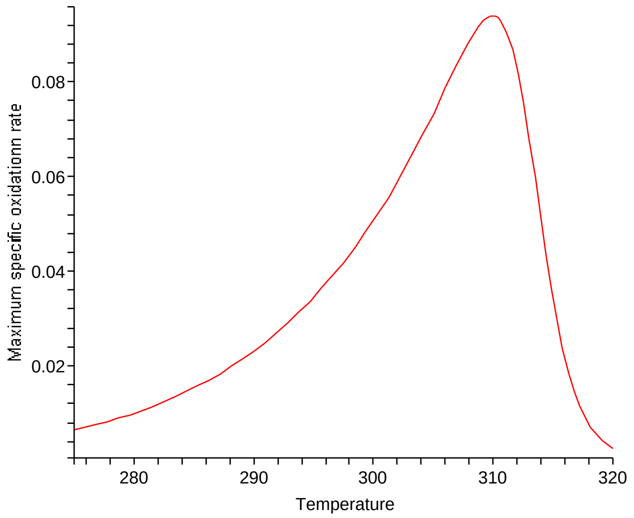

Figure 1: Figure for question (5i) showing the maximum specific oxidation rate as a function of temperature.

(b) $T = 309.9\,(\mathrm{K})$.