School of Mathematics & Applied Statistics

MATH11: Mathematics: Applied Mathematical Modelling 1

Assignment Week 4 _ Solutions

Spring 2004

1. How long will it take \$1000

   (a) to earn \$100 at 15% simple interest?

   (b) to accumulate to at least \$1200 at 13.5% simple interest?

   Solution.
   To answer this questions we use the simple interest formula

   $$S_n = S_0\left(1+\frac{np}{100}\right).$$

   (a) We have $S_0 = 100$, $S_n = (1000+100) = 1100$ and $p = 15$. Thus

   $$\begin{aligned}1100 &= 1000\left(1+\frac{15n}{100}\right)\\ \Longrightarrow n &= \frac{100}{15}\left(\frac{1100}{1000}-1\right)\\ &= \frac{2}{3}\end{aligned}$$

   Therefore it will take **8 months** for \$1000 to earn \$100 in interest at 15% simple interest.

   (b) We have $S_0 = 100$, $S_n = 1200$ and $p = 13.5$. Thus

   $$\begin{aligned}1200 &= 1000\left(1+\frac{13.5n}{100}\right)\\ \Longrightarrow n &= \frac{100}{13.5}\left(\frac{1200}{1000}-1\right)\\ &= \frac{40}{27}\end{aligned}$$

   Therefore it wil take $\mathbf{1\frac{13}{27}}$ **years** (or $\mathbf{533\frac{1}{3}}$ **days**) for \$1000 to accumulate to \$1200 at 13.5% simple interest.

2. A cash discount of 4% is given if a bill is paid 30 days in advance of its due date. What is the highest simple interest rate at which you can afford to borrow money in order to take advantage of the cash discount?

   Solution.
   To answer this questions we use the simple interest formula

   $$S_n = S_0\left(1+\frac{np}{100}\right).$$

   Suppose that the bill is for \$$b$. The cash discount of 4% means that we only pay \$$0.96b$. Thus $S_0 = 0.96b$.
   The maximum amount of money that we want to pay back is $S_n = b$, for $S_n > b$ there is no point in taking out a loan to 'save' 4%. The loan is for 30 days, thus $n = \frac{30}{360} = \frac{1}{12}$. Thus

   $$\begin{aligned}b &= 0.96b\left(1+\frac{p}{1200}\right)\\ \Rightarrow p &= 1200\left(\frac{1}{0.96}-1\right)\\ &= 50.\end{aligned}$$

   Therefore the highest simple interest rate at which you can afford to borrow money in order to t advantage of the cash discount is **50%**.

3. What is the

   (a) interest rate compounded monthly that is equivalent to 10.08% compounded yearly?

   (b) interest rate compounded every two months that is equivalent to 12% compounded quarterly?

   (c) interest rate compounded monthly that is equivalent to 5% compounded every half-year?

   (See question 6 in chapter 3.7.2 for the definition of *equivalent* interest rates).

   Solution.
   Two nominal rates of interest with difference frequencies of conversion are said to be *equivalent* if t yield the same accumulated value at the end of one year (and hence, at the end of any number of yea (Question 6 from chapter 3.7.2).

   To answer these questions we use the compound interest formula

   $$S_n = S_0\left(1+\frac{\alpha p}{100}\right)^n.$$

   (a) Over one-year 10.08% compounded yearly produces an amount

   $$\begin{aligned}S_1 &= S_0\left(1+\frac{1\times 10.08}{100}\right)^1\\ &= 1.1008S_0.\end{aligned}$$

   Over one-year p% compounded monthly produces an amount

   $$S_{12} = S_0\left(1\frac{p}{12\times100}\right)^{12}.$$

   For the interest rates to be *equivalent* we require

   $$\begin{aligned}S_1 &= S_{12}\\ 1.1008S_0 &= S_0\left(1\frac{p}{12\times100}\right)^{12}\\ \Longrightarrow p &= 1200\left[(1.008)^{1/12}-1\right]\\ &= 9.64\%\end{aligned}$$

   Therefore an interest rate of **9.64%** compounded monthly is equivalent to 10.08% compounded yea

   (b) Over one-year 12.00% compounded quarterly produces an amount

   $$S_4 = S_0\left(1+\frac{12.00}{4\times100}\right)^4$$

   Over one-year p% compounded every two-months produces an amount

   $$S_6 = S_0\left(1\frac{p}{6\times100}\right)^6.$$

   For the interest rates to be *equivalent* we require

   $$\begin{aligned}S_4 &= S_6\\ S_0\left(1+\frac{12.00}{4\times100}\right)^4 &= S_0\left(1\frac{p}{6\times100}\right)^6\\ \Longrightarrow p &= 600\left[\left(1+\frac{12}{400}\right)^{2/3}-1\right]\\ &= 11.94\%\end{aligned}$$

   Therefore an interest rate of **11.94%** compounded every two-months is equivalent to 12% compoun quarterly.

(c) Over one-year 5.00% compounded every half-year produces an amount

$$S_2 = S_0\left(1 + \frac{5}{2 \times 100}\right)^2$$

Over one-year p% compounded monthly produces an amount

$$S_{12} = S_0\left(1_{\frac{p}{12\times 100}}\right)^{12}.$$

For the interest rates to be *equivalent* we require

$$\begin{aligned} S_2 &= S_{12} \\ S_0\left(1 + \frac{5}{2 \times 100}\right)^2 &= S_0\left(1_{\frac{p}{12\times 100}}\right)^{12} \\ \implies p &= 1200\left[\left(1 + \frac{5}{20}\right)^{1/6} - 1\right] \\ &= 4.95\% \end{aligned}$$

Therefore an interest rate of **4.95%** compounded every month is equivalent to 5% compounded every half-year.

4. To prepare for early retirement, a self-employed consultant deposits \$5500 into a retirement saving plan each year, starting on her 31st birthday. When she is 51, she wishes to draw out 30 equal annual payments. What is the size of each withdrawal, if interest was compounded annually at 12% for the first ten years. compounded annually at 10% for the next ten-year perioid, and compounded annually at 11% for the 30-year retirement period?

Solution.
To answer this question we use the annuities formula

$$y_n = \left(1 + \frac{\alpha p}{100}\right)^n\left(y_0 + \frac{100R}{\alpha p}\right) - \frac{100R}{\alpha p}$$

During the first ten-years this is an annuity with interest compounded yearly at 12% and a regular deposit of \$5500. At the end of ten years the amount of money in the annuity will be

$$y = y_{10} = \left(1 + \frac{12}{100}\right)^{10}\left(y_0 + \frac{550,000}{12}\right) - \frac{5500000}{12}$$

During the second ten-years this is an annuity with interest compounded yearly at 10% and a regular deposit of \$5500. The initial amount of money in the account is the value we have just determined, i.e. y. Thus at the end of the second ten years the accumulated sum in the annuity is

$$z = y_{10} = (1.01)^{10}\left(y + \frac{550,000}{10}\right) - \frac{550,000}{10}$$

During the next thirty years the consultant draws a yearly sum from her account. The money remaining in the account gains interest compounded yearly at 11%. This is an annuity but instead of putting money into the account ($+R$) you are withdrawing money from the account (replace R in the annuities formula by $-R$). Alternatively, you can think of it as putting negative money into the annuity (replace R in the annuities formula by $-R$). Finally, you can think of this situation as being equivalent to a scenario in which a customer gives the bank a loan for the value of y_{10} — use the mortgage formula. After thirty years we want no money in the account. Thus

$$\begin{aligned} y_n &= \left(1 + \frac{\alpha p}{100}\right)^n\left(z - \frac{100R}{\alpha p}\right) + \frac{100R}{\alpha p} \\ y_{30} &= 0 \\ &= (1.11)^{30}\left(y_{10} - \frac{100R}{11}\right) + \frac{100R}{11} \end{aligned}$$

We now reach the tricky part (!). Does the customer open the account on her 31st birthday by putt
\$5500 into it? Or does she open the account on her 31st birthday and make the first deposit of \$5500
her 32nd birthday?

If you think the former you obtain

$$\begin{aligned} y &= 113600.2080 \\ z &= 382305.5183 \\ R &= \mathbf{43974.53874} \end{aligned}$$

If you think the latter you obtain

$$\begin{aligned} y &= 96518.04288 \\ z &= 337998.7813 \\ R &= \mathbf{38878.17410} \end{aligned}$$

5. Dr. Susan Calvin buys a piece of land worth \$40,000 by paying down \$10,000 down and then tak
out a loan for \$30,000. The loan will be retired with quarterly payments over 15 years with a quarte
compounded interest rate of 8%. Find her equity at the end of nine years. See questions 2 & 3
section 3.7.5.

Solution.
The first step is to calculate the quarterly repayments. The loan repayment formula is

$$D_n = \left(1 + \frac{\alpha p}{100}\right)^n\left(D_0 - \frac{100R}{\alpha p}\right) + \frac{100R}{\alpha P}.$$

We have $D_0 = 30,000$, $\alpha = \frac{1}{4}$, $n = 15 \times 4 = 60$, $D_{60} = 0$, $p = 8$. Thus

$$\begin{aligned} R &= \frac{\alpha p D_0}{100}\left(1 + \frac{\alpha P}{100}\right)^n \cdot \frac{1}{\left(1 + \frac{\alpha p}{100}\right)^n - 1}. \\ &= \frac{30,000}{50}\left(1 + \frac{1}{50}\right)^{60} \cdot \frac{1}{\left(1 + \frac{1}{50}\right)^{60} - 1}. \\ &= 863.0389749. \end{aligned}$$

The second step is to calculate the outstanding principle after nine years. From question 2 in chapter 3
the outstanding principle after k repayments is given by

$$P_k = D_0\left(1 + \frac{\alpha p}{100}\right)^k - R\frac{\left(1 + \frac{\alpha p}{100}\right)^k - 1}{\frac{\alpha p}{100}}.$$

We have $k = 4 \times 9 = 36$. Thus the outstanding principle is

$$\begin{aligned} P_{36} &= 30000\left(1 + \frac{1}{50}\right)^{36} - 863.0389749\frac{\left(1 + \frac{1}{50}\right)^{36} - 1}{\frac{1}{50}} \\ &= 16323.45494 \end{aligned}$$

The third step is to calculate the equity at the end of nine years. From question 3 in chapter 3.7.5
learn that

$$\text{Original Loan} = \text{outstanding principle} + \text{equity}$$

Thus the equity at the end of nine years is

$$\begin{aligned} \text{equity}_{36} &= 30000 - 16323.45494 \\ &= 13676.54506. \end{aligned}$$

This is the equity in the *loan*. As Dr. Carver put down a \$10,000 deposit before taking out a loan
equity in the *land* is

$$10000 + 13676.54506 = \mathbf{23676.54506}.$$

6. Consider the following map
$$x_{n+1} = \frac{27rx_n^2\left(1-x_n\right)}{16}$$

(a) Show that if $0 \le r \le 4$ and $0 \le x_n \le 1$ then $0 \le x_{n+1} \le 1$.

(b) Show that there is only one fixed point ($x^* = 0$) for $0 \le r < \frac{64}{27}$, two fixed points when $r = \frac{64}{27}$ and three fixed points for $\frac{64}{27} \le r \le 4$. Give a formulae for the new pair $x^*_\pm$.

Solution.

(a) It is clear that if $0 \le x \le 1$ then the minimum value of the function
$$y = \frac{27rx^2\left(1-x\right)}{16}$$
is given by $y\left(0\right) = y\left(1\right) = 0$, To find the maximum value we must find the location of the turning points of this function and determine if they are local maxima or local minima.
$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{27r}{16}\left(2x - 3x^2\right)$$
$$\frac{\mathrm{d}y}{\mathrm{d}x} = 0 \Longrightarrow x = 0 \quad \text{or} \quad x = \frac{2}{3}.$$
We use the second-derivative test to determine if the points $x = 0$ and $x = \dfrac{2}{3}$ are local minima or maxima.
$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = \frac{27r}{16}\left(2 - 6x\right).$$
$$\left.\frac{\mathrm{d}^2y}{\mathrm{d}x^2}\right|_{x=0} = \frac{27r}{16} > 0 \quad \text{if } r > 0$$
$$\left.\frac{\mathrm{d}^2y}{\mathrm{d}x^2}\right|_{x=\frac{2}{3}} = \frac{27r}{16}\left(-2\right) < 0 \quad \text{if } r > 0.$$
Thus the point $x = 0$ is a local minima whilst the point $x = \frac{2}{3}$ is a local maxima. The maximum value of the function y over the range $0 \le x \le 1$ is therefore given by
$$y_{\max} = y\left(\frac{2}{3}\right) = \frac{27r}{16}\frac{4}{9}\frac{1}{3} = \frac{r}{4}.$$
Hence
$$y_{\max} < 1 \Longrightarrow r < 4.$$
Thus we have shown that if $0 \le x = x_{n+1} \le 1$ and $0 < r < 4$ then $0 \le y = x_{n+1} \le 1$.

(b) Fixed points occur when
$$x_{n+1} = x_n = x.$$
Thus
$$x = \frac{27rx^2\left(1-x\right)}{16}$$
$$0 = \frac{27rx^2\left(1-x\right)}{16} - x$$
$$0 = x\left[\frac{27rx\left(1-x\right)}{16} - 1\right]$$
Thus the fixed points are
$$x = 0$$
and the solution(s) of the equation
$$0 = \frac{27rx\left(1-x\right)}{16} - 1$$
This is a quadratic equation
$$027rx^2 - 27rx + 16.$$
The solutions of this are
$$x_\pm = \frac{27r \pm \sqrt{\left(27r\right)^2 - 4\left(27r\right)16}}{54r} = \frac{27r \pm \sqrt{27r\left(27r-64\right)}}{54r}$$
If $27r - 64 < 0$ the solution $x_\pm$ is complex and there is only one fixed point ($x = 0$).
If $27r = 64$ the solution $x_\pm = \frac{1}{2}$ and there are two fixed points ($x^* = 0$ and $x_\pm = \frac{1}{2}$).
If $27r - 64 > 0$ the solution $x_\pm$ is real and there are three fixed point ($x = 0$ and $x_\pm$).

7. In a remote region in Canada, the dynamics of a fly population has been studied and found to satisfy difference equation
$$x_{n+1} = 11 - 0.01x_n^2,$$
where $x_n (> 0)$ is the fly population density at generation n,

(a) Determine the fixed point(s) of this model.

(b) Assume that the initial fly population density is 30. By drawing a cobweb diagram determines w happens to the fly population after a very long time (i.e. as $n \to \infty$).

Solution.

(a) Fixed points occur when
$$x_{n+1} = x_n = x.$$
Thus
$$x = 11 - 0.01x^2$$
$$0 = 0.01x^2 + x - 11$$
$$x = \frac{-1 \pm \sqrt{1^2 - 4\left(0.01\right)\left(-11\right)}}{2\left(0.01\right)}$$
$$x = 10 \quad \text{and} \quad x = -110.$$

(b) They key to the cobweb diagram is sketching the the function
$$y = 11 - 0.01x^2.$$
In particular you need to show that the y-intercept occurs when $x = +\sqrt{1100} \approx 33.17$. Thus initial condition is to the left of the intercept $x \approx 33.17$. Cobwebbing shows that if $x_0 = 30$ then
$$\lim_{n\to\infty} x\left(t\right) = 10.$$