School of Mathematics & Applied Statistics

MATH11: Mathematics Applied Mathematical Modelling 1

Assignment Week 2

_ Solutions

Spring 2004

1. Give the orders of of the following difference equations and state whether they are linear, nonlinear, autonomous or non-autonomous.

   (a) $nx_{n+2} + 3n^2 x_n = x_{n-1} + 2$

   (b) $x_{n-1} + \cosh(x_n) = 2$

   Solution.
   Order.
   If we denote P and Q to be the largest and smallest subscripts on the variable that occur in the difference equation, then the **order** of the difference equation is given by $P - Q$.

   In (a) the order is $(n+2) - (n-1) = 3$.
   In (b) the order is $(n) - (n-1) = 1$.

   Linear or non-linear?.
   A difference equation is said to be linear if each of the states $\ldots, x_{n-2}, x_{n-1}, x_n, x_{n+1}, x_{n+2}, \ldots$ appearing in the equation are present linearly. Otherwise it is termed **non-linear**.

   In (a) nx_{n+2}, $3n^2 x_n$ and $x_{n-1} + 2$ are all linear terms. Thus the equations is linear.

   In (b) x_{n-1} and 2 are linear but $\cosh(x_n)$ is a non-linear term. Thus the equation is non-linear.

   Autonomous or non-autonomous?.
   If a difference equation has n or k (or whatever subscript is used for the successive time steps) terms present explicitly in the equation, then we refer to it as an non-autonomous difference equation. Otherwise it is called **autonomous**.

   In (a) the subscript is n. This appears in the equation as nx_{n+2} and $3n^2 x_n$. Thus the equation is non-autonomous.

   In (b) the subscript is n. This does not appear in the equation. Thus the equation is autonomous.

2. Consider the difference equation

$$y_k = k y_{k-1}, \quad k = 1, 2, 3 \ldots$$

   with initial condition $y_0 = 1$.

   (a) Calculate y_1, y_2, y_3, y_4 and make a guess at the "closed-form" solution of y_k.

   (b) Verify that your formula satisfies the difference equation and the initial condition.

   Solution

   (a) $y_1 = 1, y_2 = 2, y_3 = 6, y_4 = 24$.
   This sequence looks like $y_k = k!$

   (b) If $y_k = k!$ then $y_{k-1} = (k-1)!$ Consider the RHS of the equation

$$\begin{aligned} k y_{k-1} &= k\,(k-1)! \\ &= k! \\ &= y_k \\ &= \text{LHS}. \end{aligned}$$

   Furthermore $y_0 = 0! = 1$ which is the initial condition.

3. Solve the following difference equations to obtain solutions in "closed form".

   (a) $x_n - 2x_{n-1} = 0$

   (b) $x_n = x_{n-1} + 3$

   (c) $x_n + x_{n-1} = n$

   (Hint: Arithmetic-Geometric Series $\displaystyle\sum_{k=1}^{n} (-1)^{n-k} k = \frac{1}{4}(2n+1) - \frac{1}{4}(-1)^n$)

   Solution.
   The solution to the equation

$$x_n = a x_{n-1} + b\,(n)$$

   is

$$x_n = a^n x_0 + \Sigma_{p=1}^{n} b\,(p)\, a^{n-p}.$$

   (a) $x_n - 2x_{n-1} = 0$. We have

$$\begin{aligned} a &= 2, \\ b\,(n) &= 0. \end{aligned}$$

   Thus the solution is

$$x_n = 2^n x_0, \qquad \text{where } x_0 \text{ is the initial condition.}$$

   (b) $x_n = x_{n-1} + 3$. We have

$$\begin{aligned} a &= 1, \\ b\,(n) &= 3. \end{aligned}$$

   Thus the solution is

$$\begin{aligned} x_n &= x_0 + \Sigma_{p=1}^{n} 3, \\ &= x_0 + 3n \qquad \text{where } x_0 \text{ is the initial condition.} \end{aligned}$$

   (c) $x_n + x_{n-1} = n$ we have

$$\begin{aligned} a &= -1, \\ b\,(n) &= n. \end{aligned}$$

   Thus the solution is

$$\begin{aligned} x_n &= (-1)^n x_0 + \Sigma_{p=1}^{n} (-1)^{n-p} p \\ &= (-1)^n x_0 + \frac{1}{4}(2n+1) - \frac{1}{4}(-1)^n \qquad \text{(using the Hint)}, \quad = \left(x_0 - \frac{1}{4}\right)(-1)^n + \frac{1}{4}(2n+1 \end{aligned}$$

4. Consider the problem of modelling patient flow in a department of geriatric medicine. Each day following activities occur:

   - A number of new patients are admitted to the department for acute care.
   - A fraction, α, of the current patients are treated and discharged.
   - A fraction, β, of the current patients, unfortunately, die.
   - A fraction of the current patients, γ, are transferred to another section.

   (a) Write down a **word** equation that defines this problem.

(b) Write down, formally, the difference equation that describes the above scenario. Define **all** variables and explain your terms.

Solution

(a)

$$\left\{\begin{array}{c}\text{Change in}\\ \text{patient numbers}\end{array}\right\} = \left\{\begin{array}{c}\text{new}\\ \text{patients}\end{array}\right\} - \left\{\begin{array}{c}\text{patients}\\ \text{discharged}\end{array}\right\} - \left\{\begin{array}{c}\text{patients}\\ \text{died}\end{array}\right\} - \left\{\begin{array}{c}\text{patients}\\ \text{transferred}\end{array}\right\}$$

(b) Let P_d and P_{d-1} be the number of patients in the department on days d and $d-1$ respectively.
Let α, β and γ be the fraction of patients that are discharged, die and transferred respectively.
Let N be the number of new patients arriving at the department.
Then

$$\begin{aligned} P_{d+1} - P_d &= N - \alpha P_d - \beta P_d - \gamma P_d \\ \Rightarrow P_{d+1} &= N + (1 - \alpha - \beta - \gamma)\, P_d. \end{aligned}$$

5. Imagine this scenario, if you will. Economic rationalism has taken hold of your workplace and it's time to renegotiate your contract. Knowing a thing or two about maths, you make the following proposal. "Boss, I've been far too greedy. But I've come to my senses, after reading *Animal Farm*, and propose a new pay scale. Starting tomorrow, I would like you to pay me two cents..." "*It's a deal*" "...raised to the power of the number of days..." "*Sign here!*" "...the commencement of my new..." "*Next!* "...contract."

Day one, you are paid 2c (2 raised to the power of one). Day two, 4c (2 squared). Day three, 8c (2^3). Day four, 16c (2^4). Day five, 32c. For week one, you take home 62c.

(a) How much do you take home in week two?

(b) How much to you take home in week three?

(c) How much do you take home in week four?

Based on an article by Jeremy Chunn that appeared in Mens Style *Summer 2003*

Solution. The sequence $2, 4, 8, 16, 32, 64\ldots$ is a *Geometric Progression* with $k = 2$ and $a = 2$. Thus the sum of the first n term is

$$\begin{aligned} s_n &= \frac{k\,(a^n - 1)}{a - 1} \\ &= \frac{2\,(2^n - 1)}{1} \\ &= 2\,(2^n - 1)\,. \end{aligned}$$

There are five working days in the first week. So the take home pay is

$$\begin{aligned} s_5 &= 2\left(2^5 - 1\right) \\ &= 2\,(32 - 1) \\ &= 62c. \end{aligned}$$

(a) The take home pay in week two is the amount of money earnt in the first ten working days minus the money earnt in the first week.

$$\begin{aligned} \text{Pay in week 2} &= s_{10} - s_5 \\ &= 2\left(2^{10} - 1\right) - 62 \\ &= 1984c. \\ &= \$19.84 \end{aligned}$$

(b) The take home pay in week three is the total amount of money earnt in the first 15 working d
minus the money earnt in the first and second week.

$$\begin{aligned} \text{Pay in week 3} &= s_{15} - s_{10} \\ &= 2\left(2^{15} - 1\right) - 2\left(2^{10} - 1\right) \\ &= 63488c. \\ &= \$634.88 \end{aligned}$$

(c) The take home pay in week four is the total amount of money earnt in the first 20 working days mi
the money earnt in the first three weeks.

$$\begin{aligned} \text{Pay in week 4} &= s_{20} - s_{15} \\ &= 2\left(2^{20} - 1\right) - 2\left(2^{15} - 1\right) \\ &= 2031616c. \\ &= \$20316.16 \end{aligned}$$