School of Mathematics & Applied Statistics

MATH111: Mathematics Applied Mathematical Modelling 1

Assignment Week 12

Spring 2004

Student Name: ______________________________ *Student Number:* ________________

FULL WORKING is to be shown for all solutions.
Untidy or badly set out work will not be marked and will be recorded as unsatisfactory.
This assignment is to be handed in during your tutorial in Week 13

1. The transmission dynamics of a disease in a population is represented by the equation

$$\frac{\mathrm{d}I}{\mathrm{d}t} = \beta I\left(1 - \frac{I}{K}\right) - \alpha I,$$

where I is the number of infected individuals in the population, β denotes the transmission coefficient, K is the total population size and α the recovery rate. Assume that $K > 0$, $\alpha > 0$ and $\beta > 0$.

   (a) Find the steady-state solutions of this model and determine their stability as a function of the recovery rate α).

   (b) Hence, or otherwise, determine the condition for the disease to be eradicated.

   (c) Draw a steady-state diagram for this model treating the recovery rate, α, as the control parameter. Indicate stable and unstable steady-state solutions using solid and dashed lines respectively.

2. (a) A population is governed by the differential equation

$$x' = x\left(e^{3-x} - 1\right).$$

   Find all steady-state solutions and determine their stability.

   (b) A fraction $p\,(0 < p < 1)$ of the population in part (a) is removed in unit time so that the population size is governed by the differential equation

$$x' = x\left(e^{3-x} - 1\right) - px.$$

   For what values of p is there a stable positive equilibrium?

School of Mathematics & Applied Statistics **MATH111: Applied Mathematical Modelling 1**
Assignment Week 12
Spring 2004 Submission Receipt

Student Name: ______________________________ *Student Number:* ________________

Tutorial Class: __________ *Date Submitted:* ________________ *Tutor Initials:* ________________

3. A population of sandhill cranes (*Grus canadiensis*) has been modelled by a logistic equation with carrying capacity of 194,600 members and intrinsic growth rate 0.0987year^{-1}. Find the critical harvest rate for which constant yield harvesting will drive the population to extinction, and find the equilibrium population size under constant yield harvesting of 3000 birds per year. You may quote appropriate results from your lecture notes.

4. (Tricky) Consider the model (Smith, 1963)

$$x' = \frac{rx(K-x)}{K+ax}$$

subjected to constant yield harvesting

$$x' = \frac{rx(K-x)}{K+ax} - h. \tag{1}$$

(a) Consider the equation

$$dx^2 + ex + f = 0, \quad d > 0 \quad f > 0.$$

Explain why the roots of this equation, should they exist, are positive if and only if $e < 0$.

(b) Show that the steady-states of equation (1) are given by

$$rx^{*^2} + (ah - rK)x^* + hK = 0$$

Explain why harvesting is only sustainable if

- $ah - rK < 0$.
- $a^2h^2 - 2r(a+2)Kh + r^2K^2 \geq 0$.

(c) Suppose that $r = K = 1$ and $a = 2$. Find the maximum sustainable value of the harvesting parameter h.