10 The period-doubling cascade to chaos

10.1 Behaviour in the logistic map when $0 < r < 1+\sqrt{6}$

In section 4 we found that the logistic model (16) has two fixed points

$$x_0^* = 0, \qquad x_1^* = 1 - \frac{1}{r}. \tag{83}$$

When $0 < r < 1$ the only fixed-point of biological interest is the trivial solution (x_0^*). This is stable for $0 < r < 1$ and unstable for $1 < r \le 4$. The second fixed point (x_1^*) only makes biological sense for $1 < r \le 4$. It is stable for $1 < r < 3$ and unstable for $3 < r \le 4$. When $r > 3$ both fixed points are unstable. In section 8.1 we showed that for $3 < r \le 4$ there is period-2 orbit which is stable for $3 < r < 1+\sqrt{6}$ and unstable for $1+\sqrt{6} < r \le 4$.

This behaviour is best appreciated using a diagram on which one plots the fixed points, as functions of r, and also indicates their stability. Figure 9 (a) shows such a diagram. Stability is indicated by showing stable orbits with solid lines and unstable ones as dashed lines. Observe that the two points comprising the period-2 orbit 'sandwich' the now unstable period-1 point (x_1^*) that generated them. Usually the unstable solution branches are not shown, resulting in a figure similar to figure 9 (b).

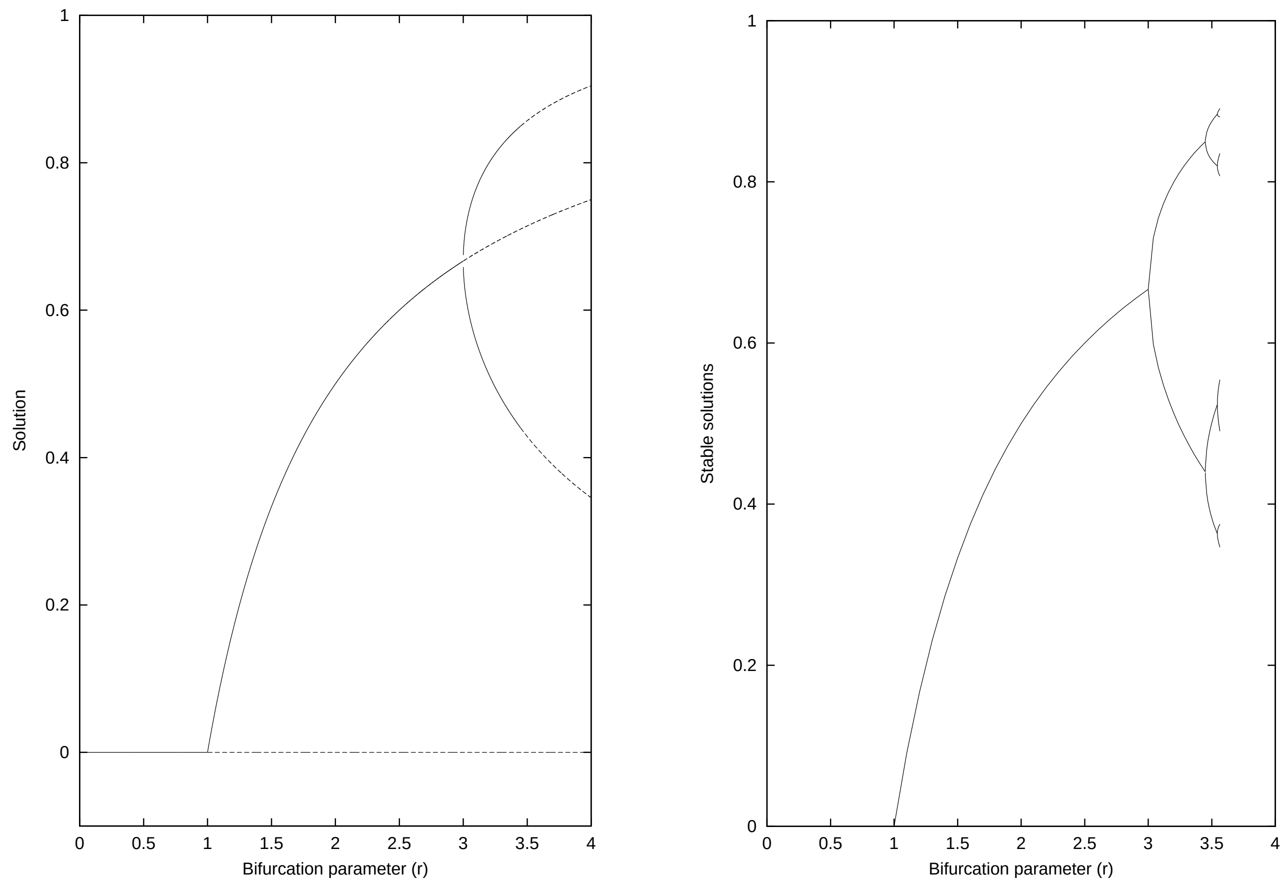
When $r = 3$ the eigenvalue of the non-trivial fixed point decreases through the value $\boxed{\lambda = -1}$. This corresponds to a $\boxed{\text{pitchfork bifurcation}}$.

For $3 < r < 1+\sqrt{6}$ the solutions x_t simply oscillates between the two points which are the intersections of a vertical line through the r-value. Numerically the map $f^2(u)$ converges to one of these two points. What happens when $r > 1+\sqrt{6}$?

10.2 Behaviour in the logistic map when $1+\sqrt{6} < r < r_\infty (r_\infty \sim 3.57)$ — the period-doubling cascade to chaos

When $r = 3$ the fixed point x_1^* destabilised, producing two points that comprised the period-2 solution. As r increases from $r = 3$, the eigenvalues λ at A and C in figure 8 decrease, eventually passing through $\lambda = -1$. At this point ($r = 1+\sqrt{6}$) the period-2 solutions become $\boxed{\text{unstable}}$. The mechanism that produced the period-2 solution from the period-1 solution is repeated: each of the period-2 points is destabilised, producing two additional solutions. A $\boxed{\text{period-4}}$ solution therefore appears at the point $r = 1+\sqrt{6}$. Thus if $r_4 < r < r_8$, where r_8 is the bifurcation value to a period-8 solution, x_t exhibits a period-4 solution with the values given by the intersection of the curve of equilibrium states with the vertical line through the r-value in figure 9 (b). The period-4 points are found by solving the equation $\boxed{x^* = f^4(x^*)}$. The function $f^4(u)$ tends to one of the four period-4 points unless for some n, $f^n(u)$ equals one of the points of period 1 or 2.

The eigenvalue of the period-4 solution decrease as r increases and eventually pass through the value $\boxed{\lambda = -1}$. At this point the stable period-4 solutions become $\boxed{\text{unstable}}$ and a stable



(a) (b)

Figure 9: Steady-state diagrams for the logistic model. Figure (a) shows the locus of the stable and unstable branches for the period-1 and period-2 solutions. Period doubling at $r = 3$; a period 2 orbit is born as a fixed point becomes unstable. Stable solutions are denoted by solid lines and unstable solutions by dashed lines. (b) Stable solutions for the logistic model as r passes through bifurcation values. At each bifurcation the previous state becomes unstable. The sequence of stable solutions have periods $2, 2^2, 2^3, \ldots$.

period-8 solution is formed. The period-8 solution is stable if $r_8 < r < r_{16}$. Table 1 shows the first few values of r_{2^n}. These numbers show that the distance between bifurcations in r-space become progressively smaller as n increases. As a result, the higher the value of n, the smaller the interval of r over which the period 2^n solution is stable. As the parameter r increases through the region $3 \leq r \leq r_\infty (r_\infty \sim 3.57)$ there is a hierarchy of solutions of period 2^n for every n, and associated with each, is a parameter interval in which it is stable. Orbits of period 2^n are successively supplanted by stable solutions of period 2^{n+1} and this transition occurs as the eigenvalue of the former decreases through $\lambda = -1$ (which is the condition for a *period doubling bifurcation*). The lower period solutions remain but are no longer stable.

The sequence $1 \to 2 \to 2^2 \to 2^3 \to \cdots$ is known as an infinite cascade of periodic orbits. At the critical value $r_\infty \approx 3.57$ all periodic solution of period 2^n are unstable and 'chaos

r_2	3
r_4	3.4494897428
r_8	3.5440903596
r_{16}	3.5644072661
r_{32}	3.5687594195
r_{64}	3.5696916098
r_∞	3.5699456...

Table 1: The values of r at which an orbit of period 2^n becomes stable in the logistic map $u_{n+1} = ru_n(1 - u_n)$.

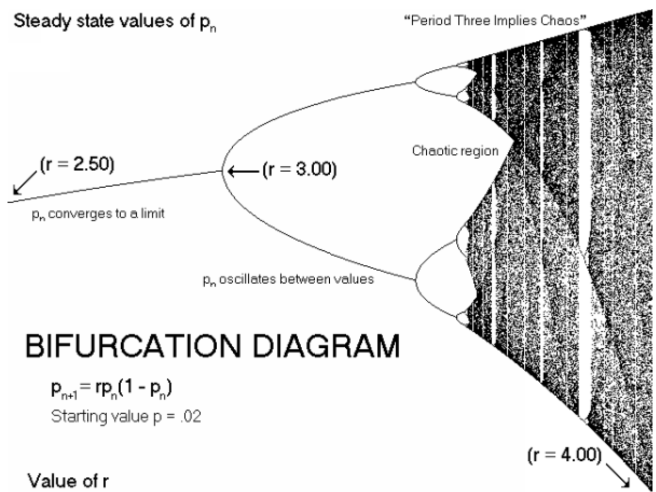


Figure 10: Stable solutions for the logistic model (16). This picture is typical of discrete models which exhibit period doubling and eventually chaos and the subsequent path through chaos. Picture downloaded from `www.pha.jhu/ldb/seminar/logdiffeqn.html`.

sets in'. This process in which an orbit of period-2^n successively lose stability to an orbit of period-2^{n+1}, ending at a limiting value at which all periodic solutions are unstable is known as the *period doubling route to chaos*.

10.3 Behaviour in the logistic map when $r_\infty < r \leq 4$

Is r_∞ the end of the story? What happens when $r_\infty < r \leq 4$?[2]

As the parameter r is increased from r_∞ there are regions where the solution is not chaotic but is instead periodic. These regions of periodicity are known as 'windows' and for $r_\infty < r < 4$ parameter windows of periodicity are interlaced with windows of aperiodicity. The interlacing of periodicity and aperiodicity is apparent from figures 10 & 11.

If the regions of periodicity are blow-up it is seen that each window contains it's own period-doubling sequence. For instance, the period-doubling cascade associated with the period-3 window will be $\boxed{3 \to 2 \times 3 \to 2^2 \times 3 \to 2^m \times 3 \to r_{3,\infty}}$, where $r_{3,\infty}$ is the accumulation point at which the period-3 period-doubling cascade becomes chaotic. In fact for $r > r_\infty$ there is

[2]The material in this section will not be tested. If you find it interesting you can do a project on it.

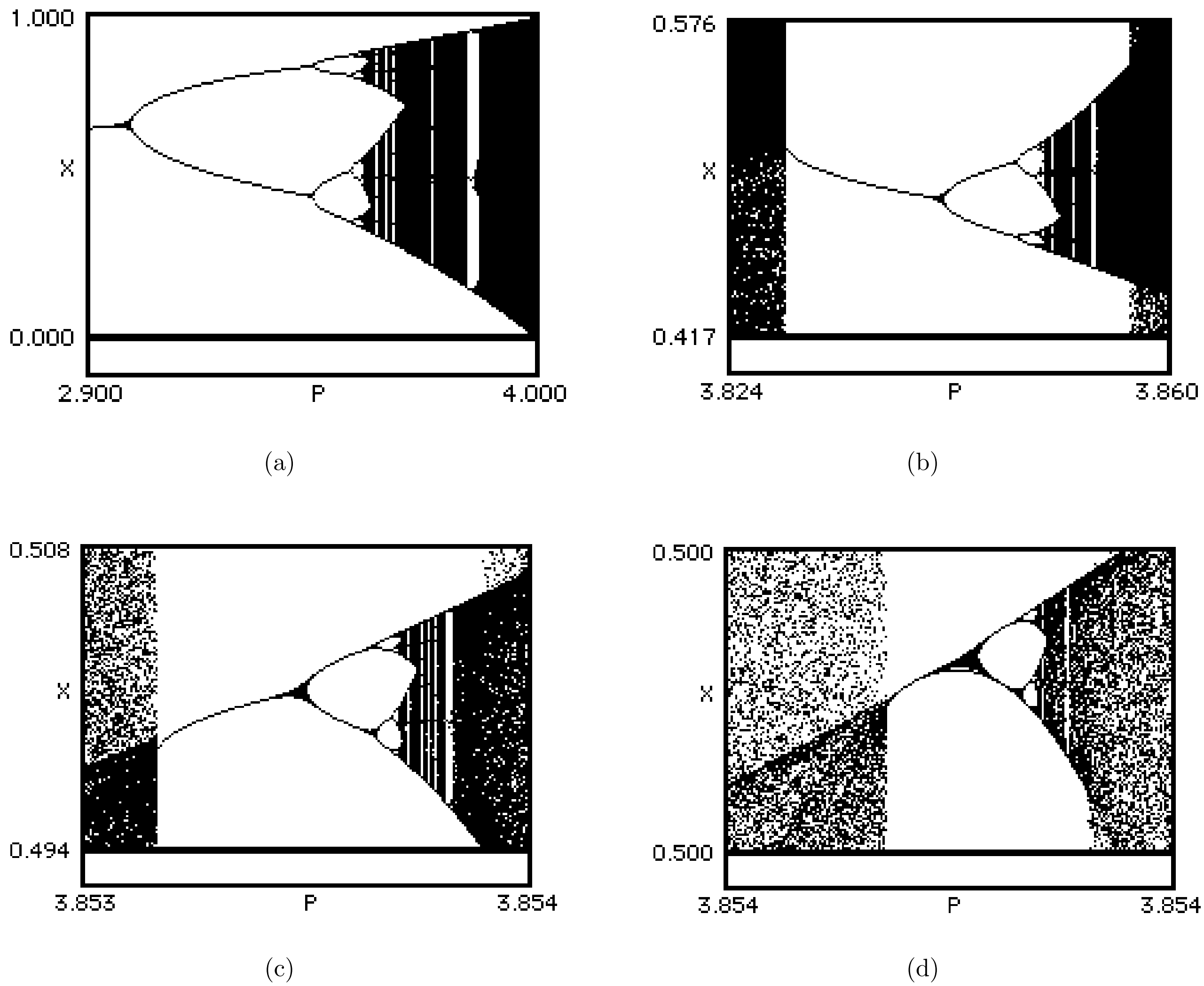


Figure 11: Stable states of the logistic model (16) and enlargement of various 'windows of stability' within it. Pictures downloaded from `www.computermusic.ch/files/articles/Chaos%2CSelf-Similarity/Chaos.html`

a periodic window of base period for any k (with k odd) and an associated period-doubling cascade $\boxed{k \rightarrow 2 \times k \rightarrow 2^2 \times k \rightarrow 2^m \times k \rightarrow r_{k,\infty}}$ ending in a chaotic region. This behaviour is illustrated in figures 10 & 11. In particular the sequence of aperiodicity - periodicity - aperiodicity is shown in the enlargements of figure 11 (a). There exist an infinite number of windows with a finite width. The period-3 window around $r \sim 3.84$ is the largest window.

Although we have concentrated here on the logistic model (16) the phenomena of period-doubling cascades to chaos and of windows surrounded by regions of aperiodicity is typical of difference equation models with the dynamics like (1) and schematically illustrated in figure 1.