On Crystal Sets

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Abstract

We are interested in 2-*crystal sets* and *protocrystal sets* in which every difference between distinct elements occurs zero or an even number of times.

We show that several infinite families of such sets exist. We also give non-existence theorems for infinite families. We find conditions to limit the computer search space for such sets. We note that search for 2-*crystal sets* $(n; k_1, k_2)$, $k = k_1 + k_2$ even, in a set of size n, immediately cuts the search space for two circulant weighing matrices with periodic autocorrelation function zero from 3^{2n} to 2^{2n-k}. We show that $2-(2n; 4, 1)$, for n odd, can only exist when 7|n and conjecture that 2-$(2n; q^2, 1)$ crystal sets will only exist when $q^2 + q + 1$ is a prime and $q^2 + q + 1|n$.

Keywords algorithm, weighing matrix, periodic autocorrelation function, zero positions, crystal set, protocrystal set, supplementary difference set, difference sets

1 Introduction

Two sequences with elements $0, \pm 1$, very small periodic or non-periodic autocorrelation function and small cross correlation function are of considerable interest in signal processing.

Two such sequences with zero periodic or non-periodic autocorrelation function are also used to form weighing matrices.

This paper concentrates on searching for the zeros of such sequences, this is called *crystallization* and the zero positions form *crystal sets*. This paper gives conditions on crystal sets and gives algorithms for their construction preparatory to searching for weighing matrices.

2 Definitions and Preliminaries

2.1 Protocrystal Sets and Crystal Sets

Difference sets [1] and *supplementary difference sets* (sds) [11, 12] and their applications have been extensively studied in the past.

We now study two more relaxed sets *protocrystal sets* and *crystal sets* which can sometimes be used to form difference sets and supplementary difference sets.

Definition 1 Let K be subset of size k, written as $(n; k; \mu)$protocrystal set, of a set of n elements, V. K will be called a *protocrystal set* if in the totality (multiset), written as Λ, of all the differences between all distinct elements in the subset, K, has an even number of even elements, $|\Lambda| = \mu$. Since $\mu = k(k-1)$ we will omit μ and write $(n; k)PCset$.

Lemma 1 *If n is odd the number of elements of Λ which are even equals the number of elements which are odd, that is $\frac{k(k-1)}{2}$: if n is even the number of odd elements in Λ is even and the number of even elements is even but they may not be equal.*

Proof. If n is odd and the protocrystal set has k elements, then the differences $(a_i - b_i)$(mod n) and $(b_i - a_i)$(mod n) both occur in Λ. Hence each difference d and $n - d$ occurs: one is even and the other is odd so the number of even and odd elements in Λ is the same. The total number of elements in Λ is $k(k-1)$: hence in this case the number of even elements is $\frac{k(k-1)}{2}$.

However if n is even, $(a_i - b_i)$(mod n) even(or odd) $\Longrightarrow$ $(b_i - a_i)$(mod n) even(or odd respectively). Hence the number of even elements in Λ is even and the number of odd elements is also even but they may not equal each other. □

Corollary 1 *Suppose n is odd. We write Λ_i for the number of elements in Λ for $k \equiv$ 0, 1, 2, or 3 (mod 4) respectively. Then we see Λ_0 and Λ_1 have an even number of even elements: but Λ_2 and Λ_3 have an odd number of even elements.*

Hence crystal sets can be made only by having two sets of size k_i, i = 0 and/or 1 (mod 4) or by having two sets of size k_i, i = 2 and/or 3 (mod 4).

Example 1 Consider $C = \{0, 1, 3, 10, 12\}$(mod 13). This has differences $(a_i - b_i)$(mod 13) where $a_i \neq b_i$, a_i, $b_i \in C$. As both $(a_i - b_i)$(mod 13) and $(b_i - a_i)$(mod 13) both occur, and since 13 is odd the number of even (and odd) elements in Λ will be the same 10. C is a $(13; 5)PC$ (proto-crystal) set.

a_1/b_1	0	1	3	10	12
0	$*$	1	3	10	12
1	12	$*$	2	9	11
3	10	11	$*$	7	9
10	3	4	6	$*$	2
12	1	2	4	11	$*$

so the totality of differences (multiset) is

$$\Lambda = [1, 1, 2, 2, 2, 3, 3, 4, 4, 6, 7, 9, 9, 10, 10, 11, 11, 11, 12, 12].$$

Definition 2 Two 2-$(n; k_1, k_2; \mu)$ subsets C_1, and C_2, of a set V of size n which have size k_1 and k_2, respectively, will be said to be *crystal sets* when Λ the totality (multiset) of all the differences from both of the subsets has each element occurring zero or an even number of times.

By counting the differences we see $\mu = k_1(k_1 - 1) + k_2(k_2 - 1)$ so we usually write $2 - (n; k_1, k_2)$ crystal sets.

Example 2 Consider $C_1 = \{0, 1, 3, 10, 12\}$ and $C_2 = \{1, 3, 10, 12\}$ (mod 13). These have differences $(a_i - b_i)$(mod 13), as follows ,where $a_i \neq b_i,\ a_i,\ b_i \in C_j$,

a_1/b_1	0	1	3	10	12
0	$*$	1	3	10	12
1	12	$*$	2	9	11
3	10	11	$*$	7	9
10	3	4	6	$*$	2
12	1	2	4	11	$*$

a_2/b_2	1	3	10	12
1	$*$	2	9	11
3	11	$*$	7	9
10	4	6	$*$	2
12	2	4	11	$*$

so the totality of differences (multiset) is

$$\Lambda = [1, 1, 2, 2, 2, 2, 2, 2, 3, 3, 4, 4, 4, 4, 6, 6, 7, 7, 9, 9, 9, 9, 10, 10, 11, 11, 11, 11, 11, 11, 12, 12].$$

Λ has each difference an even number of times so we have 2-(13; 5, 4) crystal sets. We note $\mu = 32$.

Example 3 It is possible to have a protocrystal set that is by itself a crystal set. For example consider $\{0, 1, 2, 4\}$(mod 7).

2.2 Weighing matrices

A weighing matrix $W = W(n, k)$ is a $n \times n$ square matrix with entries $0, \pm 1$ having k non-zero entries per row and column and inner product of distinct rows equal to zero. Therefore W satisfies $WW^T = kI_n$. The number k is called the weight of W. Weighing matrices were first studied because of a statistical application in weighing experiments. Later a conjecture of Seberry Wallis, that if $n \equiv 0$(mod 4) weighing matrices, $W(n, k)$, exist for all k=0, $\cdots$, n [13] sparked further work. Further conjectures concerning weighing matrices have been studied extensively, see [10] and references therein. A well-known necessary condition for the existence of $W(2n, k)$ matrices states that if there exists a $W(2n, k)$ matrix with n odd, then $k < 2n$ and k is the sum of two squares. The two circulants construction for weighing matrices is described in the theorem below, taken from [6] and is of special interest because of its applications in signal processing.

Theorem 1 *If there exist two circulant matrices A_1, A_2 of order n, with $0, \pm 1$ elements, satisfying $A_1 A_1^t + A_2 A_2^t = kI_n$ and k is an integer, then there exists a $W(2n, k)$, given as*

$$W(2n,k) = \begin{pmatrix} A_1 & A_2 \\ -A_2^t & A_1^t \end{pmatrix} \text{ or } W(2n,k) = \begin{pmatrix} A_1 & A_2 R \\ -A_2 R & A_1 \end{pmatrix}$$

where R is the square matrix of order n with $r_{ij} = 1$ if $i + j - 1 = n$ and 0 otherwise.

In this paper we study *crystal sets* which give positions of the zeros for $W(2n, 2n - a)$ constructed from two circulant matrices of order n that is the 2-$(n; k_1, k_2)$ crystal sets. If n is odd the weight $k = k_1 + k_2$ is equal to $2n - a = x^2 + y^2$, x, y integers.

2.3 Sequences with Zero Periodic Autocorrelation Function

Given the sequence $A = \{a_1, a_2, \ldots, a_n\}$, of length n, the *non-periodic autocorrelation function* $NPAF_A(s)$ is defined as

$$NPAF_A(s) = \sum_{i=1}^{n-s} a_i a_{i+s}, \quad s = 0, 1, \ldots, n-1, \tag{1}$$

Given A as above, of length n, the *periodic autocorrelation function* $PAF_A(s)$ is defined, reducing $i + s$ modulo n, as

$$PAF_A(s) = \sum_{i=1}^{n} a_i a_{i+s}, \quad s = 0, 1, \ldots, n-1. \tag{2}$$

Two sequences, $A = \{a_1, a_2, \ldots, a_n\}$ and $B = \{b_1, b_2, \ldots, b_n\}$ both of length n which will be useful in this paper have

$$NPAF_A(s) + NPAF_B(s) = 0, \quad s = 1, 2, \ldots, n, \tag{3}$$

or

$$PAF_A(s) + PAF_B(s) = 0, \quad s = 1, 2, \ldots, n, \tag{4}$$

are said to have *zero non-periodic auto-correlation function* or *zero periodic auto-correlation function* respectively.

2.4 Trivial and Foundational Crystal Sets

We use the following notation

$|N| = n$ is odd
N the set $\{0, 1, \ldots, n\text{-}1\}$
$\emptyset$ the empty set
C a protocrystal set which is a crystal set, $|C| = k$
PC a protocrystal set
C^C = the complement of a crystal set in N,
= all the elements of N which are not in C, $|C^C| = n - k$.

Theorem 2 *Two sets which are identical (or one a shift of the other so that is its elements are formed from the first set by adding a constant modulo the size of the set) can be used as crystal sets.*

Alternatively, if PC is any protocrystal set, then $\{PC,\ PC\}$ *That is all* $2-(n;k,k)$ *exist provided* $2n-2k$ *is the sum of two squares.*

Lemma 2 $\{0\}$ *and* $\{\emptyset\}$ *are possible* $2-(n;1,0)$ *crystal sets, for* n *odd and* $2n-1$ *the sum of two squares.*

Lemma 3 *The following are always possible parameters for two crystal sets,* n *odd (i)* $\emptyset$*,* C*, are 2-(n;0,k) for* $2n-k$ *the sum of two squares; (ii)* $N/\ \{0\}$*,* C *are 2-*$(n;n-1,k)$ *for* $(n-k-1)$ *the sum of two squares; (iii)* $\{0\}$*,* C *are 2-*$(n;1,k)$ *for* $2n-1-k$ *the sum of two squares; (iv)* N*,* C *are* $2-(n;n,k)$ *for* $n-k$ *the sum of two squares.*

Remark 1 Let n be odd and C_1 and C_2 be two protocrystal sets of size k_1 and k_2 respectively. We recall from the properties of weighing matrices that C_1 and C_2 can only be $2-(n,k_1,k_2)$ crystal sets if $2n-k_1-k_2$, the number of non-zero elements, is the sum of two squares.

However it is possible that if C_1 and C_2 are not $2-(n,k_1,k_2)$, that is $2n-k_1-k_2$ is not the sum of two squares, but (i) C_1 and C_2^C could be $2-(n,k_1,n-k_2)$ or
(ii) C_2 and C_1^C could be $2-(n,k_2,n-k_1)$ or
(iii) C_1^C and C_1^C could $2-(n,n-k_1,n-k_2)$
if (i)$n-k_1+k_2$, or (ii) $n+k_1-k_2$, or (iii)(k_1+k_2), respectively are the sum of two squares.

Example 4 We note that for $n = 11$, there are are no $2-(11,2,6)$ crystal sets as $2n-k_1-k_2$ =14 is not the sum of two squares, in fact

$$k_1 = 2,\ k_2 = 6 \text{ neither are the sum of two squares,}$$
$$2n-k_1-k_2 = 14,\ n-k_1+k_2 = 15\text{: neither are the sum of two squares,}$$
$$n+k_1-k_2 = 7,\ k_1+k_2 = 8 \text{ and 8 is two squares}$$
$$k_1 = 2 \neq k_2 = 6.$$

This tells us that we only need to search for sets of size 11-2 and 11-6, as in (iii).

Lemma 4 *If* $A = \{a_1, a_2, \cdots, a_{k_1}\}$ *and* $B = \{b_1, b_2, \cdots, b_{k_2}\}$ mod n*,* n *odd are two crystal sets* $(n;k_1,k_2;\mu)$ *then* A^C *and* B^C *are two crystal sets.*

Proof Let $2L$ be the set of all differences from the set $N = \{0,1,2 \cdots, n-1\}$ and Λ_1 is the set of differences from A and B. Then for n odd $2L$ contains $1,\ 2,\ 3,\ \cdots,\ n-1$ $2n$ times, n odd. Hence A^C and B^C will contain each difference an even number of times. □

Remark 2 In Lemma 4 if A has k_1 elements and B has k_2 elements then A^C has $n-k_1$ elements and B^C has $n-k_1$ elements. In order to minimize any searches for crystal sets we can consider the pair $A,\ B$ or $A^C,\ B^C$ which has the $\min(k_2, n-k_1)$.

Example 5 For $n = 11$, let A, B have $(k_1,\ k_2)$ =6, 8. Hence $(n-k_1,\ n-k_2)$ =(11-6, 11-8) = (5, 3) for A^C and B^C. So we could choose to search for the crystal sets with sizes 3 and 5 knowing that if they do not exist and Λ does not have each element and even number of times, then there are no crystal sets with sizes 6 and 8. If each element does occur an even number of times then A^C and B^C will be crystal sets.

Lemma 5 *A 2-$\{n; k_1, k_2; \Lambda\}$ crystal set corresponds to even $PAF_A(j) + PAF_B(j)$ for all $j \in \Lambda$.*

Proof. Suppose the crystal set needed to give the zero positions in the first row of the circulant matrices $A = \text{circ}\{a_1, \cdots, a_n\}$ and $B = \text{circ}\{b_1, \cdots, b_n\}$ where the non-zero positions are marked $*$ which means ± 1.
Write $C = [AB]$.
Then if $i \in \Lambda$, it must occur $2\lambda_i$ times. That means that in the inner product of row 1 of C with row i of C a zero element occurs in the same $2\lambda_i$ columns of C.
Rearranging the columns of C to obtain C^* we see that row 1 and row i may be written as

$$\overbrace{00\cdots\cdots 00}^{k_1+k_2}\overbrace{**\cdots\cdots**}^{2n-k_1-k_2}$$

$$\underbrace{0\cdots 0}_{2\lambda_i}\quad\underbrace{*\cdots *}_{(k_1+k_2-2\lambda_i)}\quad\underbrace{0\cdots 0}_{(k_1+k_2-2\lambda_i)}\quad\underbrace{*\cdots *}_{(2n-2k_1-2k_2+2\lambda_i)}$$

So the inner product of row 1 of C^* and row i of C^* has even number of non-zero terms. Rearranging the columns back to C gives $PAF_A(j) + PAF_B(j)$ is even $\forall i \in \Lambda$. □

Corollary 2 *Let $n = q^2 + q + 1$, q is a prime power. Then there exists a 2-$\{n; q^2, 1; \Lambda\}$ crystal set where Λ is the elements $1, 2, \cdots, q^2, q$ each $q(q-1)$ times.*

Proof For the first row of A put the zeros in the positions given by the complement of the elements in the $(q^2 + q + 1, q + 1, 1)$ difference set(from the projective plane) in the $(q^2 + q + 1, q^2, q(q-1))$ difference set. For the first row of B make the first element 0. Now n $= q^2 + q + 1$, $k_1 = q^2$, $k_2 = 1$ and $2\lambda_i = q(q-1)\ \ \forall i$. So $PAF(i) = 2q^2 + 2q + 2 - 2q^2 - 2 + 2\lambda_i = 2q + q^2 - q = q(q+1)$. Which is always even. □

Theorem 3 *Suppose there exists a cyclic difference set with parameter (v, k, λ), λ even, v odd. Then there exists a 2-$\{v; k, 1; \Lambda\}$ crystal set where Λ is the elements $1, 2, \cdots, v-1$ each $\lambda = \frac{k(k-1)}{v-1}$ times.*

Proof. As above noting $PAF(i) = 2v - 2k - 2 + \frac{k(k-1)}{v-1}$ is even.

Remark 3 There are combinations, for example, a difference set repeated and 2-$\{v; k_1, k_2; 2\lambda\}$ sds which give similar results.

Example 6 There is a (7,4,2) difference set {0,1,2,4} which can be used to give the crystal set {0,1,2,4}⊕{0} and the first rows

0	1	2	3	4	5	6	0	1	2	3	4	5	6
0	0	0	1	0	1	1	0	1	1	−	1	−	−

which can be used in the two circulant construction.

Example 7 There is a (13,9,6) difference set {0,1,2,4,5,6,7,8,10} which can be used to give the first two rows in the two circulant construction:

0	1	2	3	4	5	6	7	8	9	10	11	12
0	0	0	1	0	0	0	0	0	1	0	1	1
0	1	−	1	1	−	−	−	−	1	1	−	1

3 Crystalization

In [7] Kotsireas and Koukouvinos have found a number of new $W(2n, 2n-5)$ weighing matrices constructed from two circulants, for n odd. They used A and B to denote the first rows of the two circulants used to construct a $W(2n, 2n-5)$ as per Theorem 1. In [7] it was observed, experimentally, that if one fixes the locations of 4 of the 5 zeros for the sequences to make a $W(2n, 2n-5)$ as shown in (5)

$$\begin{array}{rccccc} A = & 0 & 0 & a_3 & \dots & a_n \\ B = & 0 & 0 & b_3 & \dots & b_n \end{array} \tag{5}$$

then the 5th zero, for the pattern (3,2) or (2,3), can only appear in precisely the position $\frac{n+3}{2}$ in one or the other of A and B. Thus we have more generally:

Theorem 4 *The crystalization pattern (t, 1) in length n, n odd is only possible if there is a single crystal set* (n, t, Λ).

Proof The second sequence, B, with a single zero has the corresponding protocrystal set {0} which has each difference occurring an even number of times, that is zero times. Hence if there is a single crystal set (n, t, Λ), we have two sets which have Λ as the totality of differences and they are all even. □

Lemma 6 *The two sets* $C_1 = \{0,\ j,\ n-j\}$ *and* $C_2 = \{j,\ n-j\}$ *are* $2-(n; 3, 2)$ *crystal sets, n odd.*

Proof C_1 has differences $\pm j,\ \pm(n-j),\ \pm(n-2j)$ and C_2 has differences $\pm(n-2j)$ and modulo n, n odd. Thus each difference occurs an even number of times and so we have two crystal sets or 2-$(n;3,2)$ crystal sets. □

Remark 4 These can be used to give the potential zeros of the first rows of two $0,\pm1$ circulant matrices giving a $W(2n,\ 2n-5)$ for $2n-5$ the sum of two squares, n odd.

Remark 5 I Kotsiras and C Koukouvinos [7] have had great success when searching for cases where there is a total of $2n-j$ zeros, $j\equiv\ 1(\text{mod } 4)$.

3.1 Crystalization Pattern (k,ℓ) or 2-$(n;k,\ell)$ Crystal Sets

In future if the number of zeros in the sequences (first rows) A and B are both equal to ℓ we will say the sequences have pattern (ℓ,ℓ); if the number of zeros in A and B, is k and ℓ, $k>\ell$, respectively, we will say the sequences have pattern (k,ℓ) [2]. This is the same as saying the structural pattern (k,ℓ) means

$$\text{there are } k \text{ zeros in } [a_1,\ldots,a_n] \text{ and } \ell \text{ zeros in } [b_1,\ldots,b_n].$$

This pattern of the zeros has been called the $(n;k,\ell)$ *crystalization* of the zeros. The positions of the non-zero elements in any sequence has been called *the support* .
We will generalize the notion of *crystalization* as outlined in [7] by using *crystal sets.*

Theorem 5 *Suppose C_1 and C_2 are 2-$(n;k_1,k_2)$ crystal sets. Then C_1 and C_2 can be used to place the zeros for the (k_1,k_2) structural pattern for the construction of two circulant matrices which may give a weighing matrix.*

Proof. Form the totality, Λ, of the differences from the elements of the crystal sets. Suppose difference i occurs λ_i times in Λ. If the elements of the crystal sets are the zero elements of the first rows of two circulant $0,\pm1$ matrices of order n (even or odd) then the inner product of row 1 and row i will have $(2n-2(k_1+k_2)+\lambda_i)$, an even number, of entries non-zero.

This is the same as saying $PAF(A,i)+PAF(B,i)$ is even $\forall i=1,\ldots,\frac{n-1}{2}$. It must be even so the number of non-zero entries is able to be even, and so, the number of +1s and −1s can cancel to give inner product of rows k and $k+i-1$ to be zero. □

Corollary 3 *The two first rows of two circulant weighing matrices must have their zeros in the positions of crystal sets.*

3.2 Crystals Sets from Difference Sets and SDS

From Seberry Wallis [12] we see that 2-$(n;k_1,k_2;\lambda)$ sds are similar to 2-$(n;k_1,k_2)$ crystal sets except that each non-zero difference in Λ must occur the same number of times, λ, and occurs for both even and odd entries.

Theorem 6 *Suppose there exist 2-*$(n; k_1, k_2; \lambda)$ *sds (for reference see [11, 12]) with* λ *even, then they form 2-*$(n; k_1, k_2; \lambda)$ *crystal sets; and the complementary 2-*$(n; n-k_1, n-k_2; 2n-2k_1-2k_2+\lambda)$ *sds or 2-*$(n; n-k_1, n-k_2)$ *crystal sets.*

Similarly a *difference set* (n, k, λ) is a single set with each non-zero difference in Λ must occur the same number of times, λ.

Theorem 7 *Every* (n, k, λ) *difference set with* λ *even is a single* $(n, k)PC$*.*

Thus we can sometimes combine difference sets to give crystal sets:

Theorem 8 *A* (n, k_1, λ_1) *difference set together with a* (n, k_2, λ_2) *difference set, where* $\lambda_1 + \lambda_2$ *is even, gives 2-*$(n; k_1, k_2)$ *crystal sets.*

Definition 3 We call the sets $(n, n)PC$, $(n, 0)PC$, $(n, 1)PC$ and $(n, n-1)PC$, which always exist, the *trivial cases*. For convenience we will write them as $(n, \psi)PS$. We note that in all trivial cases all entries of Λ occur an even number of times.

Theorem 9 *Suppose there exists a* (n, k, λ) *difference set, n odd, then its complementary* $(n, n-k, n-2k+\lambda)$ *difference set also exists. If* λ *is odd (respectively even) then* $n-2k+\lambda$ *will be even (or odd respectively). We suppose the* (n, k, λ) *difference set, n odd, has* λ *even (if not we will use the complementary set). Then the* (n, k, λ) λ *even, difference set and any* $(n, \psi)PC$*, trivial set give 2-*$(n; k, \psi)$ *crystal sets.*

Theorem 10 *Suppose* $n = q^2+q+1$*, q a prime power. Then there exist 2-*$(q^2+q+1; q^2, 1)$ *crystal sets. If* $n = q^2+q+1$ *is a prime there exists a* $W(2(q^2+q+1), q^2+1)$*.*

Proof. For these values of q there is a projective plane of order q which gives a $(q^2+q+1, q^2, q(q-1)$ difference set. The Legendre construction [14, p9] shows there is a $\{0, \pm 1\}$ circulant matrix of order n, n odd prime, with $n-1$ non-zero elements for each row and column and inner products of rows -1. The incidence matrix of the projective plane has inner product of all its rows 1. Thus these two circulant matrices give the first rows for our 2-circulant matrices to construct the 2-$CW(2(q^2+q+1), q^2+1)$. □

Example 8 $\{1,\ 2,\ 4\}$ is a 2-$(7, 2, 1)$ difference set. So we use this to make the complementary 2-$(7, 4, 2)$ difference set and with the trivial set with 1 element we find the 2-complementary sequences:

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0 1 1 0 1 0 0 : 0 1 1 - 1 - -
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This example was shown us by I Kotsireas.

3.3 Crystalization Pattern $(3, 2)$ or $2-(n; 3, 2)$ Crystal Sets n odd

Remark 6 From Lemma 6 we see that these crystal sets exist for all odd size sets. This greatly reduces the search space in looking for $(0, \pm 1)$ with zero periodic autocorrelation function as we have cut the search space from 3^{2n-2} to 2^{2n-5}.

3.4 The partition (4,1)

Remark 7 Kotsireas and Koukouvinos [7] mentioned the possibility of the pattern (4,1) and Kotsireas provided the only known example. These results inspired us to consider the more general question of when the partition (4,1) could exist.

We note that the general pattern for 4 zeros in one set is

$$0, \underbrace{*, \cdots, *}_{j}, 0, \underbrace{*, \cdots, *}_{k}, 0, \underbrace{*, \cdots, *}_{\ell}, 0, \underbrace{*, \cdots, *}_{m},$$

where

$$n = j + k + \ell + m + 4. \tag{6}$$

□

This general arrangement means we can write the zeros as occurring at positions $x_1 = 0$, $x_2 = j+1$, $x_3 = j+k+2$, $x_4 = j+k+\ell+3$. We assume $j,\ k,\ \ell,\ m$ are all nonnegative and each is $\leq n-4$. So the differences we obtain are

$(x_i - x_j)$	0	$j+1$	$j+k+2$	$j+k+\ell+3$
0	$*$	$j+1$	$j+k+2$	$j+k+\ell+3$
$j+1$	$-j-1$	$*$	$k+1$	$k+\ell+2$
$j+k+2$	$-j-k-2$	$-k-1$	$*$	$\ell+1$
$j+k+\ell+3$	$-j-k-\ell-3$	$-k-\ell-2$	$-\ell-1$	$*$

and each must occur an even number of times that is 0 or 2 or 4 ....

Remark 8 We observe that if $j = k = \ell = m$ then Equation 6 becomes $4j = n-4$ which is not possible for $j,\ k,\ \ell,\ m$ all non negative integers when n is odd.

Remark 9 We note none of $j,\ k,\ \ell,\ m$ can be $-1 (\text{mod } n)$ as each of them is non-negative and $\leq n-4$. For any of them to be non-zero it would have to be the equivalent $(n-1)(\text{mod } n)$. This is not possible. This is exclusion by the pigeonhole principle.

Lemma 7 *Suppose n is odd. Then if the element given by $(x_i - x_j)(\text{mod } n)$ is even the element given by $(x_j - x_i)(\text{mod } n)$ will be odd (and vice versa). Hence $j+1 \neq (-j-1)(\text{ mod } n)$.*

Lemma 8 *Suppose n is odd. Then it is only possible for two of $j,\ k,\ \ell$ and m to be equal if $7|n$.*

Proof. Without any loss of generality we will write $j = k = a-1$ and $j+k+\ell+3 = b$ (that is $b = n-m-1$). Then the differences from $x_1 = 0$, $x_2 = a$, $x_3 = 2a$ and $x_b = b$ are given in the following table:

$(x_i - x_j)(\text{mod } n)$	0	a	$2a$	b
0	$*$	a	$2a$	b
a	$-a$	$*$	a	$b-a$.
$2a$	$-2a$	$-a$	$*$	$b-2a$
b	$-b$	$a-b$	$2a-b$	$*$

Those that have not already paired are:

$$2a, -2a, b, -b, b-a, a-b, b-2a, 2a-b.$$

We note $2a \neq b$ as that causes the zero difference to occur. This also occurs if $2a$ is set equal to $2a-b$.

We try setting $2a$ equal to each of the other differences in turn. We have from Lemma 7 that $2a \neq -2a$. $2a \neq b$ as this would leave the difference a and $-a$ to be paired which is not possible by Lemma 7.

Setting $2a = b-a$ gives the differences $\{2a, 3a, 2a, a, -2a, -3a, -2a, -a\}$ or just $\{3a, a, -3a, -a\}$ to be paired which implies $2|n$. etting $2a = a-b$ gives the same result.

Setting $2a = b-2a$ gives the differences $\{2a, 4a, 3a, 2a, -2a, -4a, -3a, -a\}$ or just $\{3a, 4a, -4a, -3a\}$ to be paired which impies $2|n$ or $7|n$. Setting$2a = -b$ gives the same result.

Thus, since n is odd we have the result. □

Theorem 11 *The general pattern* $(4,1)$ *described above can only exist for* n*, odd, if* n *is divisible by* 7*. This means we can only have* $\{0,1,2,4\}$ *modulo 7 or* $\{0, \alpha, 2\alpha, 4\alpha\}$ *modulo* 7α*.*

Proof. The "1" in the partition is obtained by having zeros on the main diagonal of B.

From the above array there are a total of 12 differences which arise from the first set. We consider their equality with the first, $j+1$, one by one.

Case 1 By Lemma 7, $j+1 \neq (-j-1)(\text{ mod } n)$.

Case 2 Suppose $j+1 = j+k+2$. Then $k = -1 \equiv n-1(\text{mod } n)$. This is excluded by the previous remark.

Case 3 Suppose $j+1 = j+k+\ell+3$. This is equivalent to saying $k+\ell+2 = 0$. This is also excluded as all are non-negative.

Case 4 Suppose $j+1 = k+1$. This is covered by Lemma 8.

Case 5 Suppose $j+1 = \ell+1$. This is covered by Lemma 8.

Case 6 Suppose $j+1 = -k-1$. This means $j+k \equiv -2$ and so is excluded by the pigeonhole principle and that all are non-negative.

Case 7 Suppose $j+1 = -j-k-\ell-3$. Then $j = m$, this is covered by Lemma 8.

Case 8 Suppose $j+1 = -k-\ell-2$, that is $j+k+\ell+3 = n$. Here, using Equation 6 we have $n = m+1$, which is not possible as $m = -1$ is excluded by the pigeonhole principle.

Case 9 Suppose $j+1=-\ell-1$. Then $j+\ell\equiv -2 \pmod n$ and so is excluded by the pigeonhole principle.
Case 10 Suppose $j+1=-j-k-2$. Then $2j+k+3\equiv 0$. This is not possible as j and k are non-negative.
Case 11 Suppose $j+1=k+\ell+2$. Using Equation 6 this means $j+j+3+m=n \pmod n$.

To simplify the visualization of this case we will rewrite the above array using $k+\ell=j-1$ and then use symbols to identify obviously even numbers of entries. Thus we have

$(x_i-x_j) \pmod n$	0	$j+1$	$j+k+2$	$j+k+\ell+3$
0	$*$	$j+1$	$j+k+2$	$2j+2$
$j+1$	$-j-1$	$*$	$k+1$	$j+1$
$j+k+2$	$-j-k-2$	$-k-1$	$*$	$\ell+1$
$j+k+\ell+3$	$-2j-2$	$-j-1$	$-\ell-1$	$*$

Thus we have the following, as yet unpaired differences is Λ

$$\Lambda_1=\{k+1,\ \ell+1,\ 2j+2,\ j+k+2,\ -k-1,\ -\ell-1,\ -2j-2,\ -j-k-2\}.$$

The only possibilities for $k+1$ are $2j+2$, $j+k+2$, $-\ell-1$, $-2j-2$ or $-j-k-2$. So we can have the cases:
Case 11.1 Suppose $k+1=j+k+2$, then $j=-1$ which is not possible by the pigeonhole principle.
Case 11.2 Suppose $k+1=-\ell-1$. Then $k+\ell+2=0$. This is not possible.
Case 11.3 Suppose $k+1=-2j-2$. Then $2j+k+3=0$. So $k=m$. This is covered by Lemma 8. This is not possible.
Case 11.4 Suppose $k+1=-j-k-2$. Then $j+2k+3=0$. This means $j=k$. However this is covered by Lemma 8.
Case 11.5 Suppose $k+1=2j+2$. Then, we form Λ

$$\Lambda_1=\{k+1,\ \ell+1,\ k+1,\ 3j+3,\ -2j-2,\ -\ell-1,\ -2j-2,\ -3j-3\}.$$

This means the unpaired elements are

$$\{\ \ell+1,\ 3j+3,\ -\ell-1,\ -3j-3\}.$$

This means that in order to pair them $\ell+1=3j+3$ or $\ell+1=-3j-3$. Now $\ell+1=-3j-3$ gives $3j+\ell+4=0$. Which is not possible.

The remaining pair is $\ell+1=3j+3$ or $3j=\ell-2$ or $3j\le n-6$ which is possible. Working backwards we have

$$\Lambda=\{j+1,j+1,-j-1,-j-1,2j+2,3j+3,2j+2,3j+3,-2j-2,-3j-3,-2j-2,-3j-3\}.$$

Hence the only surviving case is that of n divisible by 7. We replace $3j+3$ by $-4j-4$ to clarify the following. This means we can only have $\{0,1,2,4\}$ modulo 7 or $\{0,\alpha,2\alpha,4\alpha\}$ modulo 7α. □

We have shown that the general pattern (4,1) that is $2-(2n;4,1)$ crystal sets can only exist when $7|n$. This leads us to speculate that that the patterns $(q^2,1)$, that is $2-(2n;q^2,1)$ crystal sets, will only exist when q^2+q+1 is a prime and $q^2+q+1|n$.

4 Search Space Size Reduction in the search for Protocrystal sets.

4.1 Significance of These Results

In a naive search for crystal sets we would first decide to search for all protocrystal set sizes for $k_1,\ k_2$ from zero to n;

Next we would see that there is no need to look for $k_2 = 0$ unless k_1 is a square;

Next we note from Remark 1 we can reduce our search by only considering this remark and its consequences; the overall search is limited to $(\frac{n-1}{2})^2$ cases to establish existences (of course there will be far more considering inequivalence);

Now we see that $k_1 = k_2$ is a special case; We also see that $k_1 = 1$, $k_2 = 0$ is a special case; Lemma 6 tells us these always exist.

The search is now further reduced by applying Corollary 1.

Crystal Sets under n = 9, Universal Set= $\{0,1,2,3,4,5,6,7,8\}$								
	k_1	k_2	$2n - k_1 - k_2$	$n - k_1 + k_2$	$k_1 + k_2$	$k_1 = k_2$	$a^2 + b^2$	*Reference*
1	1	5		13			$2^2 + 3^2$	Remark 1
2	2	2		9		Y	$3^2 + 0^2$	Theorem 2
3	2	3	13				$2^2 + 3^2$	Remark 1
4	2	6	10				$1^2 + 3^2$	Remark 1
5	3	3	9			Y	$3^2 + 0^2$	Theorem 2
6	3	6	9				$3^2 + 0^2$	Remark 1
7	3	7	8				$2^2 + 2^2$	Remark 1
8	4	4	10			Y	$3^2 + 1^2$	Theorem 2
9	4	5	9				$3^2 + 0^2$	Remark 1
10	4	8	13				$3^2 + 2^2$	Remark 1
11	4	9	5				$1^2 + 2^2$	Remark 1

Table 1 n=9: Values for which k_1 and k_2 can give crystal sets

Crystal Sets under n = 11, Universal Set= $\{0,1,2,3,4,5,6,7,8,9,10\}$								
	k_1	k_2	$2n - k_1 - k_2$	$n - k_1 + k_2$	$k_1 + k_2$	$k_1 = k_2$	$a^2 + b^2$	*Reference*
1	1	5	16				$4^2 + 0^2$	Remark 1
2	2	2	18			Y	$3^2 + 3^2$	Theorem 2
3	2	3	17				$4^2 + 1^2$	Remark 1
4	2	6			8		$2^2 + 2^2$	Remark 1
5	2	7	13				$2^2 + 3^2$	Remark 1
6	3	3	16			Y	$4^2 + 0^2$	Theorem 2
7	3	6	13				$2^2 + 3^2$	Remark 1
8	3	7			10		$1^2 + 3^2$	Remark 1
9	4	4			8	Y	$2^2 + 2^2$	Theorem 2
10	4	5	13				$2^2 + 3^2$	Remark 1
11	4	8	10				$3^2 + 1^2$	Remark 1
12	4	9	9				$3^2 + 0^2$	Remark 1
13	5	5			10	Y	$1^2 + 3^2$	Theorem 2
14	5	8	9				$3^2 + 0^2$	Remark 1
15	5	9	8				$2^2 + 2^2$	Remark 1

Table 2 n=11: Values for which k_1 and k_2 can give crystal sets

We assume the set is of size n and we search for sets of size $k < n$.

5 Algorithm

```
... pseudocode
MAIN(v)
← input
if v ≥ 2
  then GENERATE SUBSETS UNDER V(v)
```

```
bool DETERMINE(Total[1000], NUM)
a[100], b[100] ← 0
i, j, k, cnt, No ← 0
for i ← 0 to NUM
     do for J ← 0 to cnt
             do if a[j] == Total[i]
                   then break
          if j == cnt
             then a[cnt + +] ← Total[i]
                  b[cnt − 1] + +
             else b[j] + +
for k ← 0 to cnt
     do if b[k]mod2 == 1
           then break
           else No + +
if No == cnt
  then return true
  else return false
```

```
SORTING(TotalSet[1000], num)
i, j, k, x ← 0
k ← num/2
while k ≥ 1
     do for i ← k to num
             do x ← TotalSet[i]
                j ← i − k
                while j ≥ 0 and x ≤ TotalSet[j]
                      do TotalSet[j + k] ← TotalSet[j]
                         j ← j − k
                TotalSet[j + k] ← x;
        k ← k/2
```

```
CRYSTALIZATION(n)
i, j, q, p, t ← 0
M ← pow(2, n − 1) + 1
 malloc SubSet[i]
 malloc Length[i]
———————————— Generate subsets under v
a, b ← 0
position ← 0
set[100] ← 0
set[position] ← 0
for i ← 0 to 2^(n−1)
      do if set[0] == 0
            then SubSet[a][b] ← set[0]
                  b ← b + 1
            else break
         for i ← 1 to position
               do SubSet[a][b] ← set[i]
                  b ← b + 1
         Length[a][0] = b
         if set[position] < n − 1
            then set[position + 1] ← set[position] + 1
                  position ← position + 1
         if position ≠ 0
            then position ← position − 1
                  set[position] ← set[position] + 1
            else break

———————————— Calculate the differences
for p ← 0 to M − 1
      do if Length[p][0] ≤ (n − 1)/2
            then if (Length[p][0] ∗ (Length[p][0] − 1)) mod 4 == 0
                     then for q ← 0 to M − 1
                              do Totality[1000] ← 0
                                 Num ← 0
                                 if (Length[q][0] ∗ (Length[q][0] − 1)) mod 4 == 0
                                    then Totality[Num] ← (SubSet[p][i] − SubSet[p][j]) mod n
SORTING(Totality, Num)
if Determine(Totality, Num)
   then Print Crystal Sets

for p ← 0 to M − 1
      do if Length[p][0] ≤ (n − 1)/2
            then if (Length[p][0] ∗ (Length[p][0] − 1)) mod 4 ≠ 0
                     then for q ← 0 to M − 1
                              do Totality[1000] ← 0
                                 Num ← 0
                                 if (Length[q][0] ∗ (Length[q][0] − 1)) mod 4 ≠ 0
                                    then Totality[Num] ← (SubSet[p][i] − SubSet[p][j]) mod n
SORTING(Totality, Num)
if Determine(Totality, Num)
   then Print Crystal Sets
for i ← 0 to M
      do free(SubSet[i])
```

6 Further Research

Prove the conjecture:

Conjecture 1 *The patterns* $(q^2, 1)$ *that is* $2-(2n; q^2, 1)$ *crystal sets will only exist when* q^2+q+1 *is a prime and* $q^2+q+1|n$.

Find further ways to cut down the search space. Find more infinite families of crystal sets.

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A More permissible values of n, k_1 and k_2

A.1 $n = 13$

Crystal Sets under n = 13, Universal Set= $\{0,1,2,3,4,5,6,7,8,9,10,11,12\}$								
	k_1	k_2	$2n - k_1 - k_2$	$n - k_1 + k_2$	$k_1 + k_2$	$k_1 = k_2$	$a^2 + b^2$	*Reference*
1	1	9	16				$4^2 + 0^2$	Remark 1
2	2	2		13		Y	$2^2 + 3^2$	Theorem 2
3	2	3			5		$1^2 + 2^2$	Remark 1
4	2	6	18				$3^2 + 3^2$	Remark 1
5	2	7			9		$3^2 + 3^2$	Remark 1
6	3	3	20			Y	$4^2 + 2^2$	Theorem 2
7	3	6	17				$4^2 + 1^2$	Remark 1
8	3	7	16				$4^2 + 0^2$	Remark 1
9	3	10	13				$2^2 + 3^2$	Remark 1
10	4	4	18			Y	$3^2 + 3^2$	Theorem 2
11	4	5	17				$4^2 + 1^2$	Remark 1
12	4	8		17			$4^2 + 1^2$	Remark 1
13	4	9	13				$2^2 + 3^2$	Remark 1
14	4	12	10				$1^2 + 3^2$	Remark 1
15	4	13	9				$3^2 + 0^2$	Remark 1
16	5	5	16			Y	$4^2 + 0^2$	Theorem 2
17	5	8	13				$2^2 + 3^2$	Remark 1
18	5	9		17			$4^2 + 1^2$	Remark 1
19	6	6		13		Y	$2^2 + 3^2$	Theorem 2
20	6	7	13				$2^2 + 3^2$	Remark 1
21	6	10	10				$1^2 + 3^2$	Remark 1
22	6	11	9				$3^2 + 0^2$	Remark 1

A.2 $n = 15$

Crystal Sets under n = 15, Universal Set= $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14\}$								
	k_1	k_2	$2n - k_1 - k_2$	$n - k_1 + k_2$	$k_1 + k_2$	$k_1 = k_2$	$a^2 + b^2$	*Reference*
1	1	5	24	19				Remark 1
2	1	8			9		$3^2 + 0^2$	Remark 1
3	1	9	20				$4^2 + 2^2$	Remark 1
4	2	2	26			Y	$5^2 + 1^2$	Theorem 2
5	2	3	25				$5^2 + 0^2$	Remark 1
6	2	6			8		$2^2 + 2^2$	Remark 1
7	2	7		20			$4^2 + 2^2$	Remark 1
8	2	10	18				$3^2 + 3^2$	Remark 1
9	2	11	17				$4^2 + 1^2$	Remark 1
10	3	3	24	15		Y		Theorem 2
11	3	6		18			$3^2 + 3^2$	Remark 1
12	3	7	20				$4^2 + 2^2$	Remark 1
13	3	10	17				$4^2 + 1^2$	Remark 1
14	3	11	16				$4^2 + 0^2$	Remark 1
15	4	4			8	Y	$2^2 + 2^2$	Theorem 2
16	4	5	21	16			$4^2 + 0^2$	Remark 1
17	4	8	18				$3^2 + 3^2$	Remark 1
18	4	9	17				$4^2 + 1^2$	Remark 1
19	4	12			16		$4^2 + 0^2$	Remark 1
20	4	13	13				$2^2 + 3^2$	Remark 1
21	5	5	20			Y	$4^2 + 2^2$	Theorem 2
22	5	8	17				$4^2 + 1^2$	Remark 1
23	5	9	16				$4^2 + 0^2$	Remark 1
24	5	12	13				$2^2 + 3^2$	Remark 1
25	5	13			18		$3^2 + 3^2$	Remark 1
26	6	6	18			Y	$3^2 + 3^2$	Theorem 2
27	6	7	17				$4^2 + 1^2$	Remark 1
28	6	10			16		$4^2 + 0^2$	Remark 1
29	6	11	13				$2^2 + 3^2$	Remark 1
30	6	14	10				$1^2 + 3^2$	Remark 1
31	6	15	9				$0^2 + 3^2$	Remark 1
32	7	7	16			Y	$4^2 + 0^2$	Theorem 2
33	7	10	13				$2^2 + 3^2$	Remark 1
34	7	11			18		$3^2 + 3^2$	Remark 1
35	7	14	9				$0^2 + 3^2$	Remark 1
36	7	15	8				$2^2 + 2^2$	Remark 1

A.3 $n = 17$

Crystal Sets under n = 17, Universal Set= $\{0,1,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16\}$								
	k_1	k_2	$2n - k_1 - k_2$	$n - k_1 + k_2$	$k_1 + k_2$	$k_1 = k_2$	$a^2 + b^2$	*Reference*
1	1	9		25			$5^2 + 0^2$	Remark 1
2	2	2		17		Y	$4^2 + 1^2$	Theorem 2
3	2	3	29				$5^2 + 2^2$	Remark 1
4	2	6	26				$5^2 + 1^2$	Remark 1
5	2	7	25				$5^2 + 0^2$	Remark 1
6	2	10		25			$5^2 + 0^2$	Remark 1
7	2	11			13		$3^2 + 2^2$	Remark 1
8	3	3		17		Y	$4^2 + 1^2$	Theorem 2
9	3	6	25				$5^2 + 0^2$	Remark 1
10	3	7			10		$3^2 + 1^2$	Remark 1
11	3	10			13		$3^2 + 2^2$	Remark 1
12	3	11	20				$4^2 + 2^2$	Remark 1
13	4	4	26			Y	$5^2 + 1^2$	Theorem 2
14	4	5	25				$5^2 + 0^2$	Remark 1
15	4	8		13			$3^2 + 2^2$	Remark 1
16	4	9		13			$3^2 + 2^2$	Remark 1
17	4	12	18				$3^2 + 3^2$	Remark 1
18	4	13	17				$4^2 + 1^2$	Remark 1
19	5	5		17		Y	$4^2 + 1^2$	Theorem 2
20	5	8			13		$2^2 + 3^2$	Remark 1
21	5	9	20				$4^2 + 2^2$	Remark 1
22	5	12	17				$4^2 + 1^2$	Remark 1
23	5	13	16				$4^2 + 0^2$	Remark 1
24	6	6		17		Y	$4^2 + 1^2$	Theorem 2
25	6	7		18			$3^2 + 3^2$	Remark 1
26	6	10	18				$3^2 + 3^2$	Remark 1
27	6	11	17				$4^2 + 1^2$	Remark 1
28	6	14			20		$4^2 + 2^2$	Remark 1
29	6	15	13				$4^2 + 3^2$	Remark 1
30	7	7	20			Y	$4^2 + 2^2$	Theorem 2
31	7	10	17				$4^2 + 1^2$	Remark 1
32	7	11	16				$4^2 + 0^2$	Remark 1
33	7	14	13				$2^2 + 3^2$	Remark 1
34	7	15		25			$5^2 + 0^2$	Remark 1
35	8	8	18			Y	$3^2 + 3^2$	Theorem 2
36	8	9	17				$4^2 + 1^2$	Remark 1
37	8	12			20		$4^2 + 2^2$	Remark 1
38	8	13	13				$3^2 + 2^2$	Remark 1
39	8	16	10				$3^2 + 0^2$	Remark 1
40	8	17	9				$3^2 + 0^2$	Remark 1

B Examples of Permissible n,k_1 and k_2

B.1 $n = 9$

Crystal Sets under $n = 9$, Universal Set= $\{0,1,2,3,4,5,6,7,8\}$			
	k_1	k_2	*Sample*
1	1	5	{0,1};{0,1,5}
2	2	2	{0,1};{0,8}
3	2	3	{0,1};{0,1,5}
4	2	6	{0,1};{0,1,2,4,5,7}
5	3	3	{0,1,2};{0,7,8}
6	3	6	{0,1,2};{0,1,2,3,5,6}
7	3	7	{0,1,3};{0,1,3,4,5,6,8}
8	4	4	{0,1,2,3};{0,1,5,7}
9	4	5	{0,1,2,3};{0,1,2,3,6}
10	4	8	{0,1,3,6};{0,1,2,3,4,5,6,8}
11	4	9	{0,1,3,6};{0,1,2,3,4,5,6,8}

B.2 $n = 11$

Crystal Sets under $n = 11$, Universal Set= $\{0,1,2,3,4,5,6,7,8,9,10\}$			
	k_1	k_2	*Sample*
1	1	5	{0};{0,1,2,4,7}
2	2	2	{0,1};{0,10}
3	2	3	{0,1};{0,1,6}
4	2	6	{0,1};{0,1,2,4,5,8}
5	2	7	{0,1};{0,1,2,3,4,6,7}
6	3	3	{0,1,3};{0,1,9}
7	3	6	{0,1,2};{0,1,3,6,7,9}
8	3	7	{0,1,2};{0,1,2,3,5,7,10}
9	4	4	{0,1,2,4};{0,2,9,10}
10	4	5	{0,1,2,3};{0,1,2,3,7}
11	4	8	{0,1,2,3};{0,1,2,3,5,6,7,9}
12	4	9	{0,1,3,5};{0,1,2,3,5,7,8,9,10}
13	5	5	{0,1,2,3,4};{0,1,5,7,8}
14	5	8	{0,1,2,3,5};{0,1,3,4,6,7,8,10}
15	5	9	{0,1,2,3,5};{0,1,2,3,4,5,6,8,9}

B.3 $n = 13$

Crystal Sets under $n = 13$, Universal Set$= \{0,1,2,3,4,5,6,7,8,9,10,11,12\}$			
	k_1	k_2	*Sample*
1	1	9	{0};{0,1,2,3,4,5,7,9,10}
2	2	2	{0,1};{0,12}
3	2	3	{0,1};{0,6,7}
4	2	6	{0,1};{0,1,2,3,5,7}
5	2	7	{0,1};{0,1,2,4,5,7,11}
6	3	3	{0,1,2};{0,1,12}
7	3	6	{0,1,2};{0,1,2,3,4,8}
8	3	7	{0,1,2};{0,1,2,3,5,6,10}
9	3	10	{0,1,4};{0,1,2,3,4,6,7,8,9,11}
10	4	4	{0,1,2,3};{0,1,4,9}
11	4	5	{0,1,2,3};{0,1,2,3,8}
12	4	8	{0,1,2,3};{0,1,2,3,4,7,8,10}
13	4	9	{0,1,2,3};{0,1,2,3,4,6,7,8,12}
14	4	12	{0,1,3,9};{0,1,2,3,4,5,6,7,8,9,10,11}
15	4	13	{0,1,4,6};{0,1,2,3,4,5,6,7,8,9,10,11,12}
16	5	5	{0,1,2,3,4};{0,1,2,3,12}
17	5	8	{0,1,2,3,4};{0,1,2,3,5,6,8,11}
18	5	9	{0,1,2,3,4};{0,1,2,3,5,6,7,9,10}
19	6	6	{0,1,2,3,4,5};{0,1,2,3,4,12}
20	6	7	{0,1,2,3,4,5};{0,1,2,3,4,5,9}
21	6	10	{0,1,2,3,4,5};{0,1,2,3,4,5,7,8,9,11}
22	6	11	{0,1,2,3,6,8};{0,1,3,4,6,7,8,9,10,11,12}

B.4 $n = 15$

Crystal Sets under $n = 15$, Universal Set= $\{0,1,2,3,4,5,6,7,8,9,10,11,12,13,14\}$			
	k_1	k_2	*Sample*
1	1	5	{0};{0,1,5,6,10}
2	1	8	{0};{0,1,2,4,6,7,10,14}
3	1	9	{0};{0,1,2,3,4,5,6,8,12}
4	2	2	{0,1};{0,14}
5	2	3	{0,1};{0,1,8}
6	2	6	{0,1};{0,1,2,3,8,12}
7	2	7	{0,1};{0,1,2,5,6,9,11}
8	2	10	{0,1};{0,1,2,3,4,5,8,9,10,12}
9	2	11	{0,1};{0,1,2,3,4,6,8,9,11,12,13}
10	3	3	{0,1,2};{0,1,14}
11	3	6	{0,1,2};{0,1,2,4,6,9}
12	3	7	{0,1,2};{0,1,2,5,8,10,13}
13	3	10	{0,1,2};{0,1,2,3,4,5,6,8,9,10}
14	3	11	{0,1,2};{0,1,2,3,5,6,7,8,9,12,13}
15	4	4	{0,1,2,3};{0,1,2,14}
16	4	5	{0,1,2,3};{0,1,7,13,14}
17	4	8	{0,1,2,3};{0,1,2,3,4,6,7,12}
18	4	9	{0,7,8,12};{0,1,2,3,4,5,8,10,12}
19	4	12	{0,7,8,12};{0,1,2,3,4,5,7,8,9,10,11,13}
20	4	13	{0,7,9,10};{0,1,2,3,5,6,7,9,10,11,12,13, 14}
21	5	5	{0,1,2,3,4};{0,1,2,3,14}
22	5	8	{0,1,2,3,4};{0,1,2,3,4,6,9,11}
23	5	9	{0,1,2,3,4};{0,1,2,4,5,7,8,10,12}
24	5	12	{0,3,6,9,11};{0,1,2,3,4,5,8,9,10,11,12,13, 14}
25	5	13	{0,3,6,9,11};{0,1,2,3,4,5,6,7,8,9,10,11, 14}
26	6	6	{0,1,2,3,4,5};{0,1,2,3,4,14}
27	6	7	{0,1,2,3,4,5};{0,1,2,3,4,5,10}
28	6	10	{0,1,2,3,4,6};{0,1,2,3,4,5,6,8,10,13}
29	6	11	{0,7,8,12,13,14};{0,1,2,3,5,6,8,11,12,13, 14}
30	6	14	{0,3,6,9,10,12};{0,1,2,3,4,5,6,7,8,9,10, 11,12,13}
31	6	15	{0,1,2,3,7,9};{0,1,2,3,4,5,6,7,8,9,10,11, 12,13,14}
32	7	7	{0,1,2,3,4,5,6};{0,1,2,3,4,5,14}
33	7	10	{0,1,2,3,4,5,6};{0,1,2,3,5,6,7,9,11,14}
34	7	11	{0,1,2,3,4,5,6};{0,1,2,3,4,5,7,8,9,12,13}
35	7	14	{0,1,2,4,5,8,10};{0,1,2,3,4,5,6,7,8,9,10, 11,12,13}
36	7	15	{0,1,2,4,5,8,10};{0,1,2,3,4,5,6,7,8,9,10, 11,12,13,14}

B.5 $n = 17$

Crystal Sets under $n = 17$, Universal Set= $\{0,1,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16\}$			
	k_1	k_2	*Sample*
1	1	9	{0};{0,1,2,3,4,5,6,10,12}
2	2	2	{0,1};{0,16}
3	2	3	{0,1};{0,1,9}
4	2	6	{0,1};{0,1,2,3,6,13}
5	2	7	{0,1};{0,1,2,4,5,8,14}
6	2	10	{0,1};{0,1,2,3,4,6,8,10,11,14}
7	2	11	{0,1};{0,1,2,3,4,5,6,7,9,12,14}
8	3	3	{0,1,2};{0,1,16}
9	3	6	{0,1,2};{0,1,3,7,10,11}
10	3	7	{0,1,2};{0,2,3,5,7,12,13}
11	3	10	{0,1,2};{0,2,3,5,9,10,13,14,15,16}
12	3	11	{0,1,2};{0,1,2,3,4,5,6,7,9,10,13}
13	4	4	{0,1,2,3};{0,1,2,16}
14	4	5	{0,1,2,3};{0,1,2,3,10}
15	4	8	{0,1,2,3};{0,1,2,3,5,6,7,11}
16	4	9	{0,1,2,3};{0,1,2,3,5,6,8,10,16}
17	4	12	{0,1,2,3};{0,1,2,3,4,5,6,7,9,10,12,13}
18	4	13	{0,1,2,3};{0,1,2,3,4,6,7,8,10,11,12,13,15}
19	5	5	{0,1,2,3,4};{0,1,2,3,16}
20	5	8	{0,1,2,3,4};{0,1,2,3,4,5,7,8,14}
21	5	9	{0,1,2,3,4};{0,1,2,3,5,6,10,14}
22	5	12	{0,1,2,3,4};{0,1,2,3,4,5,6,7,9,10,11,12}
23	5	13	{0,1,2,3,5};{0,1,2,3,4,6,7,8,10,11,12,14,15}
24	6	6	{0,1,2,3,4,5};{0,1,2,3,4,16}
25	6	7	{0,1,2,3,4,5};{0,1,2,3,4,5,11}
26	6	10	{0,1,2,3,4,5};{0,1,2,3,4,6,11,12,13,14}
27	6	11	{0,1,2,3,4,5};{0,1,2,3,4,6,8,12,13,15,16}
28	6	14	{0,1,2,3,4,8};{0,1,2,3,4,5,6,7,8,9,10,11,13,14}
29	6	15	{0,1,2,3,5,8};{0,1,2,3,4,5,6,7,8,9,10,12,13,14,15}
30	7	7	{0,1,2,3,4,5,6};{0,1,2,3,4,5,16}
31	7	10	{0,1,2,3,4,5,6};{0,1,2,3,4,7,9,10,11,14}
32	7	11	{0,1,2,3,4,5,6};{0,1,2,3,4,8,9,10,11,14,16}
33	7	14	{0,1,2,3,4,5,7};{0,1,2,3,4,5,6,7,9,10,11,12,14,15}
34	7	15	{0,1,2,3,5,7,15};{0,1,2,3,4,5,6,7,8,10,11,12,13,14,15}
35	8	8	{0,1,2,3,4,5,6,7};{0,1,2,3,4,5,6,16}
36	8	9	{0,1,2,3,4,5,6,7};{0,1,2,3,4,5,6,7,12}
37	8	12	{0,2,4,6,8,9,10,11};{0,1,2,3,4,5,6,7,10,11,12,14}
38	8	13	{0,1,2,3,4,5,6,8};{0,1,2,3,4,5,6,7,8,9,11,12,15}
39	8	16	{0,1,2,3,4,8,9,12};{0,1,2,3,4,5,6,7,8,9,10,11,12,13,14,16}
40	8	17	{0,1,2,3,4,8,9,12};{0,1,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16}