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An eigenvalue approach evaluating minors for weighing matrices $W(n, n - 1)$

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ABSTRACT

In the present paper we concentrate our study on the evaluation of minors for weighing matrices $W(n, n - 1)$. Theoretical proofs concerning their minors up to the order of $(n - 4) \times (n - 4)$ are derived introducing an eigenvalue approach. A general theorem specifying the analytical form of any $(n - l) \times (n - l)$ minor is developed. An application to the growth problem for weighing matrices is given.

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1. Introduction

In several applications in the mathematical sciences determinants and principal minors are required. These applications include the detection of P matrices [11], self validating algorithms, interval matrix analysis and specification of pivot patterns of matrices [10]. The direct approach for evaluating all the principal minors of a matrix A of order n by applying LU factorizations entails a remarkable time complexity of $O(2^n n^3)$ [14]. Thus analytical formulas will be useful whenever they can be derived. Generally it is very difficult to derive analytical formulas for the determinant of a given matrix or for

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its minors. When we have matrices of special structure such as Hadamard matrices [9], Vandermonde or Hankel matrices, analytical formulae can be derived.

Definition 1. A $(0, 1, -1)$ matrix $W = W(n, n - k)$, $k = 1, 2, \dots$, of order n satisfying $W^T W = WW^T = (n - k)I_n$ is called a *weighing matrix of order n and weight $n - k$* or simply a *weighing matrix*. A $W(n, n)$, $n \equiv 0 \pmod{4}$, is a *Hadamard matrix of order n* . A $W = W(n, n - k)$ for which $W^T = -W$, $n \equiv 0 \pmod{4}$, is called a *skew-weighing matrix*.

Definition 2. A $W = W(n, n - 1)$, n even, with zeros on the diagonal satisfying $WW^T = (n - 1)I_n$ is called a *conference matrix*. If $n \equiv 0 \pmod{4}$, then $W = -W^T$ and W is called a *skew-conference matrix*. If $n \equiv 2 \pmod{4}$, then $W = W^T$ and W is called a *symmetric conference matrix* and such a matrix cannot exist unless $n - 1$ is the sum of two squares: thus they cannot exist for orders 22, 34, 58, 70, 78, 94.

For more details and construction of weighing matrices the reader can refer the book by Geramita and Seberry [3].

Two important properties of the weighing matrices, which follow directly from the definition, are:

- (1) Every row and column of a $W(n, n - k)$ contains exactly k zeros;
- (2) Every two distinct rows and columns of a $W(n, n - k)$ are orthogonal to each other, which means that their inner product is zero.

For the determinant of skew symmetric matrices we have

Lemma 1. Howard [6]

- (1) If n is odd and A is a skew-symmetric matrix with real elements then $\det A = 0$. If n is even and the elements of the matrix A of order n are not from the field of characteristic 2, then $\det A = 0$.
- (2) If n is even and A is a skew-symmetric matrix with real elements then $\det A$ is $PF(A)^2$, where $PF(A)$ is the Pfaffian of A a polynomial in the entries of A .

Definition 3. Two matrices are said to be *Hadamard equivalent* or *H-equivalent* if one can be obtained from the other by a sequence of the operations:

- (1) interchange any pairs of rows and/or columns;
- (2) multiply any rows and/or columns through by -1 .

In our research we are interested in calculating the minors of weighing matrices $W(n, n - 1)$ with zeros on the diagonal, for n even. Weighing matrices with zeros on the diagonal are used in order to obtain a number of constructions for *orthogonal sets* [4]. We also evaluate the minors appearing when Gaussian Elimination with complete pivoting is applied on weighing matrices $W(n, n - 1)$. These results are used for the specification of the values that their growth factor can take.

Notation. Throughout this paper the elements of a $(0, 1, -1)$ matrix will be denoted by $(0, +, -)$. I_n stands for the identity matrix of order n , $J_{m \times n}$ and $O_{m \times n}$ stand for the $m \times n$ matrix with ones and zeros, respectively. We write $W(j)$ for the value of the minor of any $j \times j$ submatrix of the matrix W . Whenever a determinant or minor is mentioned in the work, we mean its absolute value.

2. Analytical formulas for minors of $W(n, n - 1)$

2.1. Preliminary results

- (1) For a weighing matrix $W(n, n - 1)$, since $WW^T = (n - 1)I_n$, we have that

$$\det W \equiv W(n) = (n - 1)^{\frac{n}{2}} \quad (1)$$

- (2) For $n \equiv 0 \pmod{4}$, $W(n, n - 1)$ is always equivalent to U where $U^T = -U$.
 For $n \equiv 2 \pmod{4}$, $W(n, n - 1)$ is always equivalent to U where $U^T = U$ [5].
 Thus every $W(n, n - 1)$ can be written in the following form

$$W = \begin{bmatrix} A & B \\ D & C \end{bmatrix},$$

with $B = D^T$, for $n \equiv 2 \pmod{4}$ and with $B = -D^T$, for $n \equiv 0 \pmod{4}$.

- (3) If we shift a matrix A by kl , $k \in R$, then the eigenvalues of $(A \pm kl)$ are the eigenvalues of A shifted by k i.e. $\text{eig}(A) \pm k$ [12].
- (4) For any matrix P the non-zero eigenvalues of PP^T are equal to the non-zero eigenvalues of $P^T P$.

Kravvaritis and Mitrouli [7] have shown the following proposition:

Proposition 1. *Let W be a $W(n, n - 1)$. Then all possible*

- (1) $(n - 1) \times (n - 1)$ minors of W are 0 and $(n - 1)^{\frac{n}{2}-1}$,
- (2) $(n - 2) \times (n - 2)$ minors of W are 0, $(n - 1)^{\frac{n}{2}-2}$ and $2(n - 1)^{\frac{n}{2}-2}$,
- (3) $(n - 3) \times (n - 3)$ minors of W are
 - (a) 0, $2(n - 1)^{\frac{n}{2}-3}$ or $4(n - 1)^{\frac{n}{2}-3}$ for $n \equiv 0 \pmod{4}$ and
 - (b) $2(n - 1)^{\frac{n}{2}-3}$ or $4(n - 1)^{\frac{n}{2}-3}$ for $n \equiv 2 \pmod{4}$.

3. Evaluation of minors for orthogonal matrices

If $W(n, n - k)$ is a weighing matrix then $\frac{1}{\sqrt{n-k}}W(n, n - k)$ is an orthogonal matrix. The next result for orthogonal matrices, originally proved by Szöllösi [13] allows us to connect the values of small minors of $\frac{1}{\sqrt{n-k}}W(n, n - k)$ to those of large minors. As a consequence we can do the same for $W(n, n - k)$. This connection will be explored in this section. We give a different, short proof based on eigenvalues.

Theorem 1. *Let*

$$M = \begin{bmatrix} A & B \\ D & C \end{bmatrix},$$

be an orthogonal matrix which is $(\ell, n - \ell)$ -partitioned, i.e., A is $\ell \times \ell$, C is $(n - \ell) \times (n - \ell)$, B is $\ell \times (n - \ell)$ and D is $(n - \ell) \times \ell$. Then $\det A = \det C$.

Proof. Since $MM^T = I_n = M^T M$ it follows that

$$AA^T = I_\ell - BB^T \tag{2}$$

$$C^T C = I_{n-\ell} - B^T B \tag{3}$$

Now we chase eigenvalues through Eqs. (2) and (3) with the help of the following observations:

- (1) The eigenvalues of AA^T are equal to the eigenvalues of $A^T A$ since A is a square matrix.
- (2) The eigenvalues of CC^T are equal to the eigenvalues of $C^T C$ since C is a square matrix.
- (3) The non-zero eigenvalues of BB^T are equal to the eigenvalues of $B^T B$.

If λ is an eigenvalue of AA^T with $\lambda \neq 1$, then $\lambda = 1 - \mu$, with $\mu \neq 0$ an eigenvalue of BB^T . Thus μ is an eigenvalue of $B^T B$ and hence $1 - \mu$ is an eigenvalue of $C^T C$. Finally, $\lambda = 1 - \mu$ is an eigenvalue of CC^T . Similarly, if $\rho \neq 1$ is an eigenvalue of CC^T then ρ is and eigenvalue of AA^T .

Thus the multiset of eigenvalues of $CC^T =$ multiset of eigenvalues of AA^T union $n - 2\ell$ eigenvalues 1. As a consequence we have

$$\det CC^T = \det AA^T.$$

which finishes the proof. \square

Remark 1. The result holds for unitary matrices as well. For a unitary matrix we get $\det CC^* = \det AA^*$ which leads to the same conclusion.

Following the proof of Theorem 1, considering weighing matrices in the place of orthogonal matrices, we can get the following result.

Corollary 1. *Let*

$$W(n, n - 1) = \begin{bmatrix} A & B \\ D & C \end{bmatrix}$$

be a weighing matrix partitioned as above with $\ell \leq \frac{n}{2}$. Then the lower right $(n - \ell) \times (n - \ell)$, $\ell \geq 1$, minor of W is

$$W(n - \ell) = \det C = (n - 1)^{\frac{n}{2} - \ell} \det A. \tag{4}$$

This argument takes care of a lot of things. By considering all possible upper left $\ell \times \ell$ corners A we can specify the values of all $W(n - \ell)$. As long as ℓ is small there are not many possibilities for A and computing $\det A$ or even the eigenvalues is an easy task.

3.1. Evaluation of minors for weighing matrices with zeros on the diagonal

When the weighing matrix $W(n, n - 1)$ has zeros on the diagonal it is easier to specify the possible upper left $\ell \times \ell$ corners for small values of ℓ .

- (1) When $\ell = 1$ then $\det A = 0$ and hence $\det C = 0$.
- (2) When $\ell = 2$ then

$$A = \begin{bmatrix} 0 & + \\ + & 0 \end{bmatrix} \text{ or } A = \begin{bmatrix} 0 & + \\ - & 0 \end{bmatrix}.$$

Note that in both cases $\det AA^T = 1$. Thus $\det CC^T = (n - 1)^{n-4}$, i.e. $\det C = (n - 1)^{\frac{n}{2} - 2}$.

- (3) When $\ell = 3$ then

$$A = \begin{bmatrix} 0 & + & + \\ + & 0 & + \\ + & + & 0 \end{bmatrix} \text{ or } A = \begin{bmatrix} 0 & + & + \\ - & 0 & + \\ - & - & 0 \end{bmatrix}$$

depending on whether $n \equiv 2 \pmod 4$ or $n \equiv 0 \pmod 4$. Hence $\det AA^T = 4$ or 0 which implies $\det C = 2(n - 1)^{\frac{n}{2} - 3}$ when $n \equiv 2 \pmod 4$ and $\det C = 0$ when $n \equiv 0 \pmod 4$.

- (4) When $\ell = 4$ then

$$A = \begin{bmatrix} 0 & + & + & + \\ + & 0 & + & + \\ + & + & 0 & + \\ + & + & + & 0 \end{bmatrix} \text{ or } A = \begin{bmatrix} 0 & + & + & + \\ - & 0 & + & + \\ - & - & 0 & + \\ - & - & - & 0 \end{bmatrix}$$

depending on whether $n \equiv 2 \pmod 4$ or $n \equiv 0 \pmod 4$. Hence $\det AA^T = 9$ or 1 which implies $\det C = 3(n - 1)^{\frac{n}{2}-4}$ when $n \equiv 2 \pmod 4$ and $\det C = (n - 1)^{\frac{n}{2}-4}$ when $n \equiv 0 \pmod 4$.

We now have the following Propositions. We recall that for a $W(n, n - 1)$ to exist n must be even.

Proposition 2. *Let W be a weighing matrix, $W(n, n - 1)$, of order $n > 6$, with zeros on the diagonal. Then the $(n - 1) \times (n - 1)$ minors of W are $W(n - 1) = 0$.*

Remark 2. We showed that, when we have zeros on the diagonal, we get the lowest value from those developed in Proposition 1, that is $W(n - 1) = 0$. This agrees with the result of Lemma 1, as $n - 1$ is odd and the submatrix C is skew-symmetric with real elements.

Proposition 3. *Let W be a weighing matrix, $W(n, n - 1)$, of order $n > 6$, with zeros on the diagonal. Then the $(n - 2) \times (n - 2)$ minors of W are $W(n - 2) = (n - 1)^{\frac{n}{2}-2}$.*

Remark 3. We showed that, when we have zeros on the diagonal, we get the lowest non-zero value from these developed in Proposition 1, that is $W(n - 2) = (n - 1)^{\frac{n}{2}-2}$. For the case of $n \equiv 0 \pmod 4$ this agrees with the result of Lemma 1, as $n - 2$ is even and submatrix C is skew-symmetric with real elements.

Proposition 4. *Let W be a weighing matrix, $W(n, n - 1)$, of order $n \geq 8$, with zeros on the diagonal. Then the $(n - 3) \times (n - 3)$ minors of W are $W(n - 3) = 0$ for $n \equiv 0 \pmod 4$ and $2(n - 1)^{\frac{n}{2}-3}$ for $n \equiv 2 \pmod 4$.*

Remark 4. We showed that, when we have zeros on the diagonal, we get the lowest values from these developed in Proposition 1, that is $W(n - 3) = 0$ for $n \equiv 0 \pmod 4$ and $2(n - 1)^{\frac{n}{2}-3}$ for $n \equiv 2 \pmod 4$. The zero value for $n \equiv 0 \pmod 4$ agrees with the result of Lemma 1. Since all the matrices found by removing a $\ell \times \ell$, submatrix, ℓ odd, from a skew-symmetric weighing matrix of order $n \equiv 0 \pmod 4$ while preserving the skew-symmetry, satisfy the previous sentence, we have that all the $(n - \ell) \times (n - \ell)$ minors are zero.

We can now have an analytical specification of the $W(n - 4)$ minors of $W(n, n - 1)$.

Proposition 5. *Let W be a weighing matrix, $W(n, n - 1)$, of order $n \geq 10$, with zeros on the diagonal. Then the $(n - 4) \times (n - 4)$ minors of W are $W(n - 4) = (n - 1)^{\frac{n}{2}-4}$ for $n \equiv 0 \pmod 4$ and $3(n - 1)^{\frac{n}{2}-4}$ for $n \equiv 2 \pmod 4$.*

3.2. Evaluation of minors for CP weighing matrices

Let $A = [a_{ij}] \in R^{n \times n}$. We reduce A to upper triangular form by using Gaussian Elimination (GE) operations. Let $A^{(k)} = [a_{ij}^{(k)}]$ denote the matrix obtained after the first k pivoting operations, so $A^{(n-1)}$ is the final upper triangular matrix. A diagonal entry of the final matrix will be called a pivot. Matrices with the property that no ex-changes are actually needed during GE with complete pivoting are called completely pivoted (CP) or feasible [1].

In [8] it was shown that in the upper left hand corner of a CP skew and symmetric conference matrix W , of order $n \geq 6$ the following submatrices can always occur

1. When $\ell = 1$ then

$$A = \left[\begin{matrix} + \\ \end{matrix} \right], \tag{5}$$

thus $\det A = 1$ and hence $\det C = (n - 1)^{\frac{n}{2}-1}$.

2. When $\ell = 2$ then

$$A = \begin{bmatrix} + & + \\ + & - \end{bmatrix}, \tag{6}$$

thus $\det A = 2$ and hence $\det C = 2(n - 1)^{\frac{n}{2}-2}$.

3. When $\ell = 3$ then

$$A = \begin{bmatrix} + & + & + \\ + & - & + \\ + & + & - \end{bmatrix} \quad \text{or} \quad A = \begin{bmatrix} + & + & + \\ + & - & 0 \\ + & + & - \end{bmatrix}, \tag{7}$$

Note that in both cases $\det A = 4$. Thus $\det C = 4(n - 1)^{\frac{n}{2}-3}$.

4. When $\ell = 4$ then

$$A = \begin{bmatrix} + & + & + & + \\ + & - & + & - \\ + & - & - & + \\ + & + & - & - \end{bmatrix} \quad \text{or} \quad A = \begin{bmatrix} + & + & 0 & - \\ + & - & - & - \\ + & - & + & + \\ + & + & - & + \end{bmatrix}. \tag{8}$$

Hence $\det A = 16$ or 12 , respectively, which implies $\det C = 16(n - 1)^{\frac{n}{2}-4}$ or $\det C = 12(n - 1)^{\frac{n}{2}-4}$.

So we proved, using eigenvalues, the following propositions.

Proposition 6. *Let W be a CP skew and symmetric conference matrix, $W(n, n - 1)$, of order $n > 6$. Then the $(n - 1) \times (n - 1)$ minors of W are $W(n - 1) = (n - 1)^{\frac{n}{2}-1}$.*

Proposition 7. *Let W be a CP skew and symmetric conference matrix, $W(n, n - 1)$, of order $n > 8$. Then the $(n - 2) \times (n - 2)$ minors of W are $W(n - 2) = 2(n - 1)^{\frac{n}{2}-2}$.*

Proposition 8. *Let W be a CP skew and symmetric conference matrix, $W(n, n - 1)$, of order $n > 8$. Then the $(n - 3) \times (n - 3)$ minors of W are $W(n - 3) = 4(n - 1)^{\frac{n}{2}-3}$.*

We now have analytical specification of the $W(n - 4)$ minors of a CP skew and symmetric conference matrix $W(n, n - 1)$.

Proposition 9. *Let W be a CP skew and symmetric conference matrix, $W(n, n - 1)$, of order $n > 10$. Then the $(n - 4) \times (n - 4)$ minors of W are $W(n - 4) = 16(n - 1)^{\frac{n}{2}-4}$ or $W(n - 4) = 12(n - 1)^{\frac{n}{2}-4}$.*

4. Application to the growth problem

4.1. Description of the problem

Traditionally, backward error analysis for GE [2] on a matrix $A = (a_{ij}^{(0)})$ is expressed in terms of the growth factor

$$g(n, A) = \frac{\max_{i,j,k} |a_{ij}^{(k)}|}{\max_{i,j} |a_{ij}^{(0)}|},$$

which involves all the elements $a_{ij}^{(k)}$, $k = 0, 1, 2, \dots, n - 1$ that occur during the elimination. For a CP matrix A we have

$$g(n, A) = \frac{\max\{p_1, p_2, \dots, p_n\}}{|a_{11}^{(0)}|},$$

where p_1, p_2, \dots, p_n are the pivots of A . The growth factor measures the growth of elements during the elimination.

Cryer [1] defined $g(n) = \sup\{g(n, A) \mid A \in \mathbb{R}^{n \times n}, CP\}$. The problem of determining $g(n)$ for various values of n is called the *growth problem*. Gaussian elimination will be backward stable if the growth factor is of order 1. The determination of $g(n)$ in general remains a mystery.

In [1] Cryer conjectured that “for real matrices $g(n, A) \leq n$, with equality if and only if A is a Hadamard matrix”. This conjecture became one of the most famous open problems in Numerical Analysis and has been investigated by many mathematicians. It was finally shown to be false in 1991, however its second part is still an open problem. One of the curious frustrations of the growth problem is that it is quite difficult to construct any examples of $n \times n$ matrices A other than Hadamard for which $g(n, A)$ is even close to n .

It can be proved [2] that the magnitude of the pivots appearing after the application of GE operations on a CP matrix W is given by

$$p_j = \frac{W(j)}{W(j-1)}, \quad j = 1, 2, \dots, n, \quad W(0) = 1. \tag{9}$$

So, it is obvious that the calculation of minors is important in order to study pivot structures. Thus the results of Section 3 can help us in studying the growth problem for CP weighing matrices. Its interesting to see if weighing matrices, due to their special properties can give moderate growth factor.

4.2. Specification of pivot patterns

Theorem 2. *When Gaussian Elimination is applied on a CP weighing matrix W of order n , then*

- (1) *the first four pivots are 1, 2, 2, 3 or 4,*
- (2) *the last four pivots are $\frac{n-1}{3}$ or $\frac{n-1}{4}$, $\frac{n-1}{2}$, $\frac{n-1}{2}$, $n - 1$.*

Proof. From (5)–(8) we have that

$$W(1) = 1, \quad W(2) = 2, \quad W(3) = 4, \quad W(4) = 12 \text{ or } 16$$

and, using Eq. (9), we conclude that the first four pivots of a CP weighing matrix $W(n, n - 1)$ are

$$p_1 = 1, \quad p_2 = 2, \quad p_3 = 2, \quad p_4 = 3 \text{ or } 4.$$

Furthermore, from Propositions 6–9 we have that

$$W(n - 1) = (n - 1)^{\frac{n}{2}-1}, \quad W(n - 2) = 2(n - 1)^{\frac{n}{2}-2}, \quad W(n - 3) = 4(n - 1)^{\frac{n}{2}-3},$$

$$W(n - 4) = 12(n - 1)^{\frac{n}{2}-4} \quad \text{or} \quad 16(n - 1)^{\frac{n}{2}-4}$$

and, regarding that from definition of a weighing matrix it holds $W(n) = (n - 1)^{\frac{n}{2}}$, we conclude that the last four pivots of a CP weighing matrix $W(n, n - 1)$ are

$$p_{n-3} = \frac{n - 1}{3} \quad \text{or} \quad \frac{n - 1}{4}, \quad p_{n-2} = \frac{n - 1}{2}, \quad p_{n-1} = \frac{n - 1}{2}, \quad p_n = n - 1. \quad \square$$

We can now have another proof, than the one given in [10], of the pivot structure of the weighing matrix $W(8, 7)$. As a result of the above theorem we have the following corollary.

Corollary 2. (i) The pivot patterns of the $W(8, 7)$ are

$$\left\{ 1, 2, 2, 3, \frac{7}{3}, \frac{7}{2}, \frac{7}{2}, 7 \right\} \text{ or } \left\{ 1, 2, 2, 4, \frac{7}{4}, \frac{7}{2}, \frac{7}{2}, 7 \right\}.$$

(ii) The growth factor of $W(8, 7)$ is 7.

5. An algorithm evaluating minors $W(n - \ell)$, for $\ell > 1$

As ℓ increases it is difficult to specify all possible $\ell \times \ell$ upper left corners of $W(n, n - 1)$. Next we propose an alternative technique which can lead to an algorithm evaluating with low complexity cost minors $W(n - \ell)$, $\ell > 1$.

We need the following notation. We denote with $x_{m \times n}$ the $m \times n$ block with elements x , x real, and with $X_{m \times n}$ the $m \times n$ block with the specific form of the matrix X .

Let $\tilde{x}_{\beta+1}^T$ the vectors containing the binary representation of each integer $\beta + 2^{\ell-1}$ for $\beta = 0, \dots, 2^{\ell-1} - 1$. Replace all zero entries of $\tilde{x}_{\beta+1}^T$ by -1 and define the $\ell \times \ell$ vectors $\tilde{u}_j = \tilde{x}_{2^{\ell-1}-j+1}$, $j = 1, \dots, 2^{\ell-1}$. We write U_ℓ for all the matrices with ℓ rows and the appropriate number of columns, in which the vector \tilde{u}_ℓ occurs u_ℓ times. So

$$U_\ell = \begin{matrix} \overbrace{+\cdots+}^{u_1} & \overbrace{+\cdots+}^{u_2} & \cdots & \overbrace{+\cdots+}^{u_{2^{\ell-1}-1}} & \overbrace{+\cdots+}^{u_{2^{\ell-1}}} & \tilde{u}_1 & \tilde{u}_2 & \cdots & \tilde{u}_{2^{\ell-1}-1} & \tilde{u}_{2^{\ell-1}} \\ +\cdots+ & +\cdots+ & \cdots & +\cdots+ & +\cdots+ & + & + & \cdots & + & + \\ +\cdots+ & +\cdots+ & \cdots & -\cdots- & -\cdots- & + & + & \cdots & - & - \\ \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\ +\cdots+ & +\cdots+ & \cdots & +\cdots+ & -\cdots- & + & + & \cdots & - & - \\ +\cdots+ & -\cdots- & \cdots & +\cdots+ & -\cdots- & + & - & \cdots & + & - \end{matrix} =$$

Example 1. For $\ell = 3$ and $\ell = 4$, respectively, we get

$$U_3 = \begin{matrix} \tilde{u}_1 & \tilde{u}_2 & \tilde{u}_3 & \tilde{u}_4 \\ + & + & + & + \\ + & + & - & - \\ + & - & + & - \end{matrix} \text{ and } U_4 = \begin{matrix} \tilde{u}_1 & \tilde{u}_2 & \tilde{u}_3 & \tilde{u}_4 & \tilde{u}_5 & \tilde{u}_6 & \tilde{u}_7 & \tilde{u}_8 \\ + & + & + & + & + & + & + & + \\ + & + & + & + & - & - & - & - \\ + & + & - & - & + & + & - & - \\ + & - & + & - & + & - & + & - \end{matrix}$$

In this example vectors \tilde{u}_i occurs $u_i = 1$ times.

Theorem 3. Let W be a weighing matrix, $W(n, n - 1)$, n large enough, with zeros on the diagonal. Then the $(n - \ell) \times (n - \ell)$, $\ell > 1$, minor of W is

$$W(n - \ell) = [(n - 1)^{n-\ell-2^{\ell-1}} \cdot \det M]^{\frac{1}{2}},$$

where M is a $2^{\ell-1} \times 2^{\ell-1}$ matrix of the form

$$\begin{bmatrix} n - 1 - \ell u_1 & u_1 c_{1,2} & u_1 c_{1,3} & \cdots & u_1 c_{1,2^{\ell-1}} \\ u_2 c_{1,2} & n - 1 - \ell u_2 & u_2 c_{2,3} & \cdots & u_2 c_{2,2^{\ell-1}} \\ \vdots & \vdots & \vdots & & \vdots \\ u_{2^{\ell-1}-1} c_{1,2^{\ell-1}} & u_{2^{\ell-1}-1} c_{2,2^{\ell-1}} & u_{2^{\ell-1}-1} c_{3,2^{\ell-1}} & \cdots & n - 1 - \ell u_{2^{\ell-1}} \end{bmatrix},$$

where $c_{i,j} = -\tilde{u}_i^T \cdot \tilde{u}_j$, $i, j = 1, \dots, 2^{\ell-1}$.

Proof. Take any weighing matrix, $W = W(n, n - 1)$, n large enough, with zeros on the diagonal, where $W^T = -W$ for $n \equiv 0(\text{mod } 4)$, that is the matrix is skew-symmetric, and $W^T = W$ for $n \equiv 2(\text{mod } 4)$, that is the matrix is symmetric.

If we remove the $\ell \times \ell$ principal rows and columns, then

$$W(n - \ell) = [(n - 1)^{n-\ell-2^{\ell-1}} \cdot \det M]^{1/2},$$

where M is a $2^{\ell-1} \times 2^{\ell-1}$ matrix, found by the next algorithm.

Step 1: We use any sort algorithm to sort the $\ell + 1$ to n columns of W using the topmost ℓ elements to order the sorting to obtain W_ℓ .

Step 2: We repeat the above steps on the rows of W_ℓ to obtain $W_{\ell-1}$.

Explanation of step 2: As before the first ℓ rows of W_ℓ are not altered but all other rows have been permuted so their first $\ell + 1$ elements, if in the same order will appear as groups.

Note: These sorting operations have the same effect on symmetry. If column i becomes column j , then row i will become row j . That is we have preserved the symmetry or skew-symmetry.

Now in $W_{\ell-1}$ we see a structure of the form

$$W_{\ell-1} = \left[\begin{array}{c|cccc} & \overbrace{A_{\ell \times \ell}} & \overbrace{u_1} & \overbrace{u_i} & \overbrace{u_{2^{\ell-1}}} \\ & \tilde{u}_1 \dots \tilde{u}_1 \dots \tilde{u}_i \dots \tilde{u}_i \dots \tilde{u}_{2^{\ell-1}} \dots \tilde{u}_{2^{\ell-1}} & & & \\ \left. \begin{array}{l} u_1 \\ \vdots \\ \tilde{u}_1^T \end{array} \right\} & & & & \\ \dots & & & & \\ \left. \begin{array}{l} u_i \\ \vdots \\ \tilde{u}_i^T \end{array} \right\} & & & & \\ \dots & & & & \\ \left. \begin{array}{l} u_{2^{\ell-1}} \\ \vdots \\ \tilde{u}_{2^{\ell-1}}^T \end{array} \right\} & & & & \\ & & & & C_{n-\ell \times n-\ell} \end{array} \right]$$

Step 3: Remove any ℓ rows from $W_{\ell-1}$ and the ℓ corresponding columns to make C .

Step 4: Find CC^T . We give some properties:

- (1) C is of size $(n - \ell) \times (n - \ell)$. It has a zero diagonal and other elements ± 1 .
- (2) CC^T is symmetric with $n - \ell - 1$ elements on its diagonal. The (i, j) element of CC^T is the inner product of the i th row and the j th column of C .
- (3) Consider the inner product of rows $\ell + 1$ and $\ell + 2$ of $W_{\ell-1}$ (assuming $u_1 > 0$). Then it is zero. However the inner product of the first ℓ elements of rows $\ell + 1$ and $\ell + 2$ is

$\tilde{u}_1^T \tilde{u}_1 + \tilde{u}_2^T \tilde{u}_2 + \dots + \tilde{u}_\ell^T \tilde{u}_\ell = \ell$ so the (1, 2) element of CC^T is $-\ell$. Similarly, for the first $\ell + 1, \ell + 2, \dots, \ell + u_1$ rows of $W_{\ell-1}$ and the first $1, 2, \dots, u_1$ rows of CC^T .

Thus,

$$CC^T = \begin{bmatrix} D_1 & c_{1,2J} & c_{1,3J} & \dots & c_{1,2^{\ell-1}J} \\ c_{1,2J}^T & D_2 & c_{2,3J} & \dots & c_{2,2^{\ell-1}J} \\ c_{1,3J}^T & c_{2,3J}^T & D_3 & \dots & c_{3,2^{\ell-1}J} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ c_{1,2^{\ell-1}J}^T & c_{2,2^{\ell-1}J}^T & c_{3,2^{\ell-1}J}^T & \dots & D_{2^{\ell-1}} \end{bmatrix},$$

where

$$D_i = \begin{bmatrix} n - \ell - 1 & -\ell & \dots & -\ell & -\ell \\ -\ell & n - \ell - 1 & \dots & -\ell & -\ell \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ -\ell & -\ell & \dots & -\ell & n - \ell - 1 \end{bmatrix}_{u_i \times u_i}.$$

and $c_{i,j} = -\tilde{u}_i^T \cdot \tilde{u}_j, i, j = 1, \dots, 2^{\ell-1}$.

Step 5: An algorithm for the evaluation of the determinant of matrix C

Let us call the matrix CC^T A.

1st Step: We take column u_i from columns $1, 2, \dots, u_i - 1, i = 1, 2, \dots, 2^{k-1}$ in D_i and $c_{i,jJ}$. Then, the above submatrices of CC^T are modified as

$$D_i^* = \begin{bmatrix} n - 1 & 0 & \dots & 0 & -\ell \\ 0 & n - 1 & \dots & 0 & -\ell \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & n - 1 & -\ell \\ -n + 1 & -n + 1 & \dots & -n + 1 & n - \ell - 1 \end{bmatrix}_{u_i \times u_i},$$

$$c_{i,jJ}^* = \begin{bmatrix} 0 & 0 & \dots & c_{i,j} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & c_{i,j} \\ 0 & 0 & \dots & c_{i,j} \end{bmatrix}_{u_i \times u_j}$$

2nd Step: We add rows $1, 2, \dots, u_i - 1$ to row u_i . Then,

$$D_i^\dagger = \begin{bmatrix} n - 1 & 0 & \dots & 0 & -\ell \\ 0 & n - 1 & \dots & 0 & -\ell \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & n - 1 & -\ell \\ 0 & 0 & \dots & 0 & n - u_i \ell - 1 \end{bmatrix}_{u_i \times u_i},$$

Table 1
Minors of $W(n, n - 1), n \leq 32$.

n	$W(n - 1)$	$W(n - 2)$	$W(n - 3)$	$W(n - 4)$	$W(n - 5)$
14	0	$(n - 1)^{\frac{n}{2}-2}$	$[(n - 1)^7 M_{4 \times 4}]^{\frac{1}{2}}$	$[(n - 1)^3 M_{7 \times 7}]^{\frac{1}{2}}$	-
16	0	$(n - 1)^{\frac{n}{2}-2}$	$0 = [(n - 1)^9 M_{4 \times 4}]^{\frac{1}{2}}$	$[(n - 1)^4 M_{8 \times 8}]^{\frac{1}{2}}$	-
18	0	$(n - 1)^{\frac{n}{2}-2}$	$[(n - 1)^{11} M_{4 \times 4}]^{\frac{1}{2}}$	$[(n - 1)^7 M_{7 \times 7}]^{\frac{1}{2}}$	-
20	0	$(n - 1)^{\frac{n}{2}-2}$	$0 = [(n - 1)^{13} M_{4 \times 4}]^{\frac{1}{2}}$	$[(n - 1)^8 M_{8 \times 8}]^{\frac{1}{2}}$	$0 = [(n - 1)^3 M_{12 \times 12}]^{\frac{1}{2}}$
24	0	$(n - 1)^{\frac{n}{2}-2}$	$0 = [(n - 1)^{17} M_{4 \times 4}]^{\frac{1}{2}}$	$[(n - 1)^{12} M_{8 \times 8}]^{\frac{1}{2}}$	$0 = [(n - 1)^7 M_{12 \times 12}]^{\frac{1}{2}}$
32	0	$(n - 1)^{\frac{n}{2}-2}$	$[(n - 1)^{25} M_{4 \times 4}]^{\frac{1}{2}}$	$[(n - 1)^{20} M_{8 \times 8}]^{\frac{1}{2}}$	$[(n - 1)^{12} M_{15 \times 15}]^{\frac{1}{2}}$

$$c_{i,j} \uparrow = \begin{bmatrix} 0 & 0 & \dots & c_{i,j} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & c_{i,j} \\ 0 & 0 & \dots & u_i c_{i,j} \end{bmatrix}_{\ell \times u_i}$$

3rd Step: We delete columns with $n - \ell - 1$ zero elements and the corresponding rows. We note that the remaining rows/columns are: $u_1, u_2, \dots, u_{2^{\ell-1}}$.

Then, the remaining $2^{\ell-1} \times 2^{\ell-1}$ matrix M is

$$\begin{bmatrix} n - 1 - \ell u_1 & u_1 c_{1,2} & u_1 c_{1,3} & \dots & u_1 c_{1,2^{\ell-1}} \\ u_2 c_{1,2} & n - 1 - \ell u_2 & u_2 c_{2,3} & \dots & u_2 c_{2,2^{\ell-1}} \\ \vdots & \vdots & \vdots & & \vdots \\ u_{2^{\ell-1}} c_{1,2^{\ell-1}} & u_{2^{\ell-1}} c_{2,2^{\ell-1}} & u_{2^{\ell-1}} c_{3,2^{\ell-1}} & \dots & n - 1 - \ell u_{2^{\ell-1}} \end{bmatrix},$$

So,

$$\det W = [(n - 1)^{n-\ell-2^{\ell-1}} \cdot \det M]^{\frac{1}{2}}. \quad \square$$

5.1. Implementation of the algorithm

For a given matrix $W(n, n - 1)$ we can directly specify the vectors \tilde{u}_i and the quantities u_i . Then $c_{i,j}$ are computed by simple inner products of the form $c_{i,j} = -\tilde{u}_i^T \cdot \tilde{u}_j$ requiring only $O(\ell)$ flops. Thus this algorithm achieves the evaluation of minors attaining lower complexity than a computing program. For example, the evaluation of $W(n - 3)$ for the weighing matrix $W(24, 23)$ using the proposed algorithm demands a complexity of order 4^3 in order to evaluate the determinant of the 4×4 matrix M , while a program that uses LU factorization for the direct evaluation of the determinant of CC^T would demand a complexity of order 21^3 .

6. Numerical results

We evaluated minors for weighing matrices $W(n, n - 1)$ of order up to 32. The results are given in Table 1 and are valid for minors of size $n - \ell$ reasonably smaller than n .

Note: For $n = 32, k = 5$ we expected the following $(n - 5) \times (n - 5)$ minor:

$$W(n - 5) = [(n - 1)^{11} \cdot \det M_{16 \times 16}]^{\frac{1}{2}}.$$

But carrying out the calculations in the computer for one of the over 30 million $W(32, 31)$, we found

$$W(n - 5) = [(n - 1)^{12} \cdot \det M_{15 \times 15}]^{\frac{1}{2}}.$$

0	0	0	-2	0	0	0	-2	19	0	0	0	2	0	0	0	0	0	-2
-2	0	0	0	0	0	0	0	2	23	0	0	-4	0	-2	0	-2	0	0
-2	0	0	0	0	0	0	0	2	0	23	0	-4	0	-2	0	-2	0	0
-2	0	0	0	0	0	0	0	2	0	0	23	-4	0	-2	0	-2	0	0
-2	0	0	0	0	0	0	0	2	-23	-23	-23	19	0	-2	0	-2	0	0
0	0	0	-2	0	0	0	2	0	0	0	0	-2	23	-4	0	0	0	-2
0	0	0	-2	0	0	0	2	0	0	0	0	-2	-23	19	0	0	0	-2
0	0	0	2	0	0	0	-2	0	0	0	0	-2	0	0	23	-4	0	-2
0	0	0	2	0	0	0	-2	0	0	0	0	-2	0	0	-23	19	0	-2
2	0	0	0	0	0	0	0	-2	0	0	0	0	0	-2	0	-2	23	-4
2	0	0	0	0	0	0	0	-2	0	0	0	0	0	-2	0	-2	0	23
2	0	0	0	0	0	0	0	-2	0	0	0	0	0	-2	0	-2	-23	19

Step 2: Write $v_i = \sum_{j=1}^i u_j$. We add row v_i to rows $v_{i-1} + 1, v_{i-1} + 2, \dots, v_{i-1} + u_i - 1, i = 1, 2, \dots, 8$.

19	0	0	-2	0	0	0	-2	0	0	0	0	-2	0	0	0	0	0	2
-2	23	0	-4	0	0	0	0	-2	0	0	0	0	0	-2	0	2	0	0
-2	0	23	-4	0	0	0	0	-2	0	0	0	0	0	-2	0	2	0	0
-6	0	0	11	0	0	0	0	-6	0	0	0	0	0	-6	0	6	0	0
-2	0	0	0	23	0	0	-4	-2	0	0	0	0	0	2	0	-2	0	0
-2	0	0	0	0	23	0	-4	-2	0	0	0	0	0	2	0	-2	0	0
-2	0	0	0	0	0	23	-4	-2	0	0	0	0	0	2	0	-2	0	0
-8	0	0	0	0	0	0	7	-8	0	0	0	0	0	8	0	-8	0	0
0	0	0	-2	0	0	0	-2	19	0	0	0	2	0	0	0	0	0	-2
-2	0	0	0	0	0	0	0	2	23	0	0	-4	0	-2	0	-2	0	0
-2	0	0	0	0	0	0	0	2	0	23	0	-4	0	-2	0	-2	0	0
-2	0	0	0	0	0	0	0	2	0	0	23	-4	0	-2	0	-2	0	0
-8	0	0	0	0	0	0	0	8	0	0	0	7	0	-8	0	-8	0	0
0	0	0	-2	0	0	0	2	0	0	0	0	-2	23	-4	0	0	0	-2
0	0	0	-4	0	0	0	4	0	0	0	0	-4	0	15	0	0	0	-4
0	0	0	2	0	0	0	-2	0	0	0	0	-2	0	0	23	-4	0	-2
0	0	0	4	0	0	0	-4	0	0	0	0	-4	0	0	15	0	0	-4
2	0	0	0	0	0	0	0	-2	0	0	0	0	0	-2	0	-2	23	0
2	0	0	0	0	0	0	0	-2	0	0	0	0	0	-2	0	-2	0	23
6	0	0	0	0	0	0	0	-6	0	0	0	0	0	-6	0	-6	0	11

Step 3: We expand this determinant, using the basic definition of the determinant, pivoting using the columns with a single non-zero entry. Thus we delete rows/columns 2, 3, 5, 6, 7, 10, 11, 12, 14, 16, 18, 19. Then, we have a determinant which is 23^{12} times the determinant of the 8×8 matrix M now given

19	-2	-2	0	-2	0	0	2
-6	11	0	-6	0	-6	6	0
-8	0	7	-8	0	8	-8	0
0	-2	-2	19	2	0	0	-2
-8	0	0	8	7	-8	-8	0
0	-4	4	0	-4	15	0	-4
0	4	-4	0	-4	0	15	-4
6	0	0	-6	0	-6	-6	11

Now, $\det A = 23^{12} \cdot \det M = 23^{12} \cdot 279841 = 23^{12} \cdot 23^4 = 23^{16}$, that is $\det A = (n - 1)^{n-8}$.

Thus $\det C = (n - 1)^{\frac{n}{2}-4}$ and this result agrees with the value of the $(n - 4) \times (n - 4)$ minor of a weighing matrix $W = W(n, n - 1)$ for $n \equiv 0 \pmod{4}$ given in Proposition 5.

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