AUSTRALASIAN JOURNAL OF COMBINATORICS
Volume **50** (2011), Pages 243–258

The amicable-Kronecker construction of quaternion orthogonal designs

Jennifer Seberry

Centre for Computer and Information Security Research
School of Information Technology and Computer Science
University of Wollongong, NSW 2522
Australia
jennie@uow.edu.au

Sarah Spence Adams*

Mathematics and Electrical and Computer Engineering
Franklin W. Olin College of Engineering
Olin Way, Needham, MA 02492-1245
U.S.A.
sarah.adams@olin.edu

Abstract

Recently, quaternion orthogonal designs (QODs) were introduced as a mathematical construct with the potential for applications in wireless communications. The potential applications require new methods for constructing QODs, as most of the known methods of construction do not produce QODs with the exact properties required for implementation in wireless systems. This paper uses real amicable orthogonal designs and the Kronecker product to construct new families of QODs. The proposed Amicable-Kronecker Construction can be applied to build quaternion orthogonal designs of a variety of sizes and types. Although it has not yet been simulated whether the resulting designs are useful for applications, their properties look promising for the desired implementations. Furthermore, the construction itself is interesting because it uses a simple family of real amicable orthogonal designs and the Kronecker product as building blocks, opening the door for future construction algorithms using other families of amicable designs and other matrix products.

* Corresponding author. Supported in part by National Security Agency Grant H98230-10-1-0220.

1 Introduction and Preliminaries

This section motivates the results in this paper and provides the necessary definitions and background.

1.1 Motivation

Orthogonal designs over the quaternion domain have been proposed as potential building blocks for future applications in signals processing [3, 9, 16], while orthogonal designs with quaternion elements have also been used in current applications [5]. The study of the recently proposed designs over the quaternion domain was motivated by the successful implementation of complex orthogonal designs as space-time block codes, which effectively combine space and time diversities [4, 14, 18]. The goal of the proposed future applications was to further improve performance through the additional combination of polarization diversity [6, 7, 8, 15, 24]. Since it has been shown that polarization states can be modeled by means of quaternion representations, [13], it was natural to investigate orthogonal designs over the quaternion domain.

Subsequent results [20] showed that only certain orthogonal designs over the quaternion domain will be useful in practical wireless systems. Thus, it has become increasingly important to develop additional construction techniques to generate designs more suitable for applications. In this paper, we give the first example of using real amicable orthogonal designs and a matrix product construction to build designs over the quaternion domain, though other products have been used with other types of orthogonal matrices in the past [10, 18]. The resulting quaternion designs look promising, though further simulations will be necessary to determine if they are applicable in practical systems. More importantly, we expect that the proposed idea of utilizing amicable families with matrix products will lead to improved construction techniques in the future.

1.2 Real Amicable Designs

The original definition for *orthogonal designs* proposed by Geramita, Geramita and Seberry Wallis concerned only square matrices with real commuting variables or zero entries [10, 11]:

Definition 1. An *orthogonal design*, OD, of order n and type $(s_1, s_2, \ldots, s_u)$ in commuting real variables $x_1, x_2, \ldots, x_u$, denoted $OD(n; s_1, s_2, \ldots, s_u)$, is an $n \times n$ matrix A with entries in the set $\{0, \pm x_1, \pm x_2, \ldots, \pm x_u\}$ satisfying

$$AA^T = \sum_{h=1}^{u} s_h x_h^2 I_n,$$

where I_n is the identity matrix of order n. The definition can be broadened to include

$r \times n$ matrices A where

$$A^T A = \sum_{h=1}^{u} s_h x_h^2 I_n.$$

Such rectangular orthogonal designs are denoted $OD(r; n; s_1, s_2, \ldots, s_u)$, and their columns are formally orthogonal.

Definition 2. Two square orthogonal designs A and B are said to be *amicable* if $A^T B = B^T A$ and $AB^T = BA^T$. If A and B are $r \times n$ orthogonal designs and $A^T B = B^T A$, we will also say they are *amicable.*

Example 3. Let w_h, for $h = 1, 2, \ldots, 2u$, be real variables. Define

$$W_h = \begin{bmatrix} w_h & 0 \\ 0 & -w_h \end{bmatrix} = w_h \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

Then, $W_\ell^T W_m = (w_l w_m)(I_2) = W_m^T W_\ell$ for $1 \leq \ell, m \leq 2u$. Hence, for any $l \neq m$, W_h and W_l are amicable. These matrices will be used in the case of $e = 1$ in the proof of Theorem 2.

Example 4. Let w_h, for $h = 1, 2, \ldots, 2u$, be real variables. Define

$$W_h = \begin{bmatrix} w_h & w_h \\ w_h & -w_h \end{bmatrix} = w_h \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}.$$

Then, $W_\ell^T W_m = (w_l w_m)(2I_2) = W_m^T W_\ell$ for $1 \leq \ell, m \leq 2u$. Hence, for any $l \neq m$, W_h and W_l are amicable. These matrices will be used in the case of $e = 2$ in the proof of Theorem 2.

Example 5. If w_0 and w_1 are real variables, then the orthogonal design $W_1 = \begin{bmatrix} w_0 & w_1 \\ -w_1 & w_0 \end{bmatrix}$ is amicable with all W_h introduced in Example 3 and amicable with all W_h introduced in Example 4. This will be used in the proof of Theorem 2.

Connections between amicable orthogonal designs and certain space-time block codes have been studied [21], as have generalizations including amicable complex orthogonal designs [22].

1.3 Orthogonal Designs over the Quaternions

We recall that the non-commutative quaternions $\mathcal{Q} = \{\pm 1, \pm \mathbf{i}, \pm \mathbf{j}, \pm \mathbf{k}\}$ satisfy $\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = \mathbf{ijk} = -1$. A *quaternion variable* $\mathbf{a} = a_1 + a_2\mathbf{i} + a_3\mathbf{j} + a_4\mathbf{k}$, where a_1, a_2, a_3, a_4 are real variables has a *quaternion conjugate* defined by $\mathbf{a}^Q = a_1 - a_2\mathbf{i} - a_3\mathbf{j} - a_4\mathbf{k}$. It follows that $\mathbf{a}^Q\mathbf{a} = \mathbf{a}\mathbf{a}^Q = |\mathbf{a}|^2$ is real. Given a matrix $A = (\mathbf{a}_{\ell,m})$, where $\mathbf{a}_{\ell,m}$ are quaternion variables, its *quaternion transpose* is $A^Q = (\mathbf{a}_{m,\ell}^Q)$.

We now review the definitions of quaternion orthogonal designs (QODs) originally introduced in [16], along with some new notation to help distinguish among QODs defined over real, complex, and quaternion variables (QOD-R, QOD-C, and QOD-Q, resp.).

Definition 6. A *quaternion orthogonal design on commuting real variables (QOD-R)* $x_1,\ x_2,\ \ldots,\ x_u$ of type $(s_1, s_2, \ldots, s_u)$ is an $r \times n$ matrix A with entries from $\{0, \pm\mathbf{q}_1 x_1, \pm\mathbf{q}_2 x_2,\ \ldots,\ \pm\mathbf{q}_u x_u\}$, $\mathbf{q}_h \in \mathcal{Q}$, that satisfies $A^Q A = (\sum_{h=1}^{u} s_h x_h^2)\, I_n$. This design is denoted by $A = \text{QOD-R}(r, n;\ s_1, s_2,\ \ldots, s_u)$. Similarly, a *quaternion orthogonal design on commuting complex variables (QOD-C)* $z_1, z_2, \ldots, z_u$ is an $r \times n$ matrix A with entries from the set $\{0, \pm z_1, \pm z_1^*, \pm z_2, \pm z_2^*, \ldots, \pm z_u, \pm z_u^*\}$ including possible multiplications on the left and/or right by quaternion elements $\mathbf{q} \in \mathcal{Q}$, that satisfies $A^Q A = (\sum_{h=1}^{u} s_h |z_h|^2) I_n$. This design is denoted by $A = \text{QOD-C}(r, n;\ s_1, s_2, \ldots, s_u)$. Finally, we define a *quaternion orthogonal design on non-commuting quaternion variables (QOD-Q)* $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_u$ as an $r \times n$ matrix A with entries from the set $\{0, \pm\mathbf{a}_1, \pm\mathbf{a}_1^Q, \pm\mathbf{a}_2, \pm\mathbf{a}_2^Q, \ldots, \pm\mathbf{a}_u, \pm\mathbf{a}_u^Q\}$ including possible multiplications on the left and/or right by quaternion elements $\mathbf{q} \in \mathcal{Q}$ and to satisfy $A^Q A = (\sum_{h=1}^{u} s_h |\mathbf{a}_h|^2) I_n$. This design is denoted by $A = \text{QOD-Q}(r, n;\ s_1, s_2,\ \ldots, s_u)$. In all cases, the columns of a QOD-R, QOD-C, or QOD-Q are mutually orthogonal. We can generalize these definitions to allow the design entries to be real linear combinations of the permitted variables and their quaternion multipliers, in which case we say the design is with linear processing.

We can write each complex variable z_h, $h = 1, 2, \ldots, u$, that appears within an QOD-C A as $z_h = x_{2h-1} + x_{2h}\mathbf{i}$, where the x_ℓ, $\ell = 1, 2, \ldots, 2u$, are real variables. This allows us to write the orthogonality constraint for A as

$$A^Q A = \sum_{h=1}^{u} s_h \left(|x_{2h-1}|^2 + |x_{2h}|^2\right) I_n. \tag{1}$$

Now, we notice that any entry $\mathbf{q} z_h \mathbf{q}'$ of a QOD-C A (when A is without linear processing) can be written as $\mathbf{q}(x_{2h-1} + x_{2h}\mathbf{i})\mathbf{q}'$, where $\mathbf{q}$, $\mathbf{q}'$ are quaternion elements from $\mathcal{Q}$. This entry expands to $\mathbf{q}x_{2h-1}\mathbf{q}' + \mathbf{q}x_{2h}\mathbf{i}\mathbf{q}'$, and then since quaternion elements commute with real variables, we can rewrite this entry as $\mathbf{q_{2h-1}} x_{2h-1} + \mathbf{q_{2h}} x_{2h}$, where $\mathbf{q_{2h-1}}, \mathbf{q_{2h}}$ are some quaternion elements.

So, if we permit linear processing in A, so that the entries of A are linear combinations of the terms $\mathbf{q} z_h \mathbf{q}'$, we see that a general entry of A is actually a linear combination of terms $\mathbf{q_{2h-1}} x_{2h-1} + \mathbf{q_{2h}} x_{2h}$. We can write an arbitrary such linear combination as $\sum_{h=1}^{2u} \alpha_h \mathbf{q_h} x_h$, where the α_h are appropriate real constants, the $\mathbf{q}_h$ are quaternion elements from $\mathcal{Q}$, and the x_h are real variables.

2 The Amicable-Kronecker Construction

Throughout, we let A be a QOD-C with linear processing on u complex variables $z_h = x_{2h-1} + x_{2h}\mathbf{i}$, $h = 1, 2, \ldots, u$. Such QOD-Cs can be obtained most easily using the constructions presented in Theorem 4 of [16] or Theorem 1 of [3] (additional constructions for QOD-Cs are also included in those references). The entries of the QOD-C A have the form $\sum_{h=1}^{2u} \alpha_h \mathbf{q}_h x_h$, as defined above in Section 1.3. Since each complex variable $z_h = x_{2h-1} + x_{2h}\mathbf{i}$, for $h = 1, 2, \ldots, u$, appears within every column of A, we can say that A is on real variables $x_1, x_2, \ldots, x_{2u}$, each of which appears in every column of A. Then, it is possible to rewrite A as a sum of the real variables times their *coefficient matrices* C_h defined so that:

$$A = \sum_{h=1}^{2u} x_h C_h. \tag{2}$$

A coefficient matrix will have at least one entry from $\mathcal{Q}$ (potentially with real scalar multipliers) in each row and each column, while other entries will be zero.

Definition 7. Define the *support* of a matrix to be the positions of the matrix that are nonzero.

Example 8. Let $z_1 = x_1 + \mathbf{i}x_2$ and $z_2 = x_3 + \mathbf{i}x_4$ be complex variables where x_1, x_2, x_3 and x_4 are real variables. Then, let A be the following QOD-C (with no linear processing), $A = \begin{bmatrix} z_1\mathbf{j} & z_2 \\ z_2 & z_1\mathbf{j} \end{bmatrix}$, which satisfies $A^Q A = (|z_1|^2 + |z_2|^2)I_2$, so that $s_h = 1$ for $h = 1, 2$. Then, we can decompose A into its coefficient matrices as follows:

$$\begin{aligned} A &= \begin{bmatrix} x_1\mathbf{j} + x_2\mathbf{k} & x_3 + x_4\mathbf{i} \\ x_3 + x_4\mathbf{i} & x_1\mathbf{j} + x_2\mathbf{k} \end{bmatrix} \\ &= x_1 \begin{bmatrix} \mathbf{j} & 0 \\ 0 & \mathbf{j} \end{bmatrix} + x_2 \begin{bmatrix} \mathbf{k} & 0 \\ 0 & \mathbf{k} \end{bmatrix} + x_3 \begin{bmatrix} 0 & \mathbf{1} \\ \mathbf{1} & 0 \end{bmatrix} + x_4 \begin{bmatrix} 0 & \mathbf{i} \\ \mathbf{i} & 0 \end{bmatrix} \\ &= x_1 C_1 + x_1 C_2 + x_3 C_3 + x_4 C_4. \end{aligned}$$

Notice that since $z_1 = x_1 + \mathbf{i}x_2$, the coefficient matrices C_1 and C_2 have exactly the same support; similarly, since $z_2 = x_3 + \mathbf{i}x_4$, the coefficient matrices C_3 and C_4 have exactly the same support.

Example 9. Let A be the following QOD-C with linear processing, where z_h are complex variables, z_h^* are the complex conjugates, and $A^Q A = 2(|z_1|^2 + |z_2|^2 + |z_3|^2)I_4$ so that $s_h = 2$ for $h = 1, 2, 3$:

$$A = \begin{bmatrix} z_3^* - z_1^*\mathbf{j} & z_2\mathbf{j} & z_1 + z_3\mathbf{j} & z_2 \\ -z_2^*\mathbf{j} & z_3^* - z_1\mathbf{j} & -z_2^* & z_1^* + z_3\mathbf{j} \\ -z_1^* + z_3^*\mathbf{j} & z_2 & z_3 + z_1\mathbf{j} & z_2\mathbf{j} \\ -z_2^* & -z_1 + z_3^*\mathbf{j} & -z_2^*\mathbf{j} & z_3 + z_1^*\mathbf{j} \end{bmatrix}.$$

Then, using $z_h = x_{2h-1} + x_{2h}\mathbf{i}$, for $h = 1, 2, 3$, where the x_h are real variables, we can rewrite A as:

$$A =$$
$$\begin{bmatrix} (x_5-x_6\mathbf{i})-(x_1-x_2\mathbf{i})\mathbf{j} & (x_3+x_4\mathbf{i})\mathbf{j} & (x_1+x_2\mathbf{i})+(x_5+x_6\mathbf{i})\mathbf{j} & (x_3+x_4\mathbf{i}) \\ -(x_3-x_4\mathbf{i})\mathbf{j} & (x_5-x_6\mathbf{i})-(x_1+x_2\mathbf{i})\mathbf{j} & -(x_3-x_4\mathbf{i}) & (x_1-x_2\mathbf{i})+(x_5+x_6\mathbf{i})\mathbf{j} \\ -(x_1-x_2\mathbf{i})+(x_5-x_6\mathbf{i})\mathbf{j} & (x_3+x_4\mathbf{i}) & (x_5+x_6\mathbf{i})+(x_1+x_2\mathbf{i})\mathbf{j} & (x_3+x_4\mathbf{i})\mathbf{j} \\ -(x_3-x_4\mathbf{i}) & -(x_1+x_2\mathbf{i})+(x_5-x_6\mathbf{i})\mathbf{j} & -(x_3-x_4\mathbf{i})\mathbf{j} & (x_5+x_6\mathbf{i})+(x_1-x_2\mathbf{i})\mathbf{j} \end{bmatrix}$$

$$\begin{aligned} &= x_1 \begin{bmatrix} -\mathbf{j} & 0 & \mathbf{1} & 0 \\ 0 & -\mathbf{j} & 0 & \mathbf{1} \\ -\mathbf{1} & 0 & \mathbf{j} & 0 \\ 0 & -\mathbf{1} & 0 & \mathbf{j} \end{bmatrix} + x_2 \begin{bmatrix} \mathbf{k} & 0 & \mathbf{i} & 0 \\ 0 & -\mathbf{k} & 0 & -\mathbf{i} \\ \mathbf{i} & 0 & \mathbf{k} & 0 \\ 0 & -\mathbf{i} & 0 & -\mathbf{k} \end{bmatrix} \\ &+ x_3 \begin{bmatrix} 0 & \mathbf{j} & 0 & \mathbf{1} \\ -\mathbf{j} & 0 & -\mathbf{1} & 0 \\ 0 & \mathbf{1} & 0 & \mathbf{j} \\ -\mathbf{1} & 0 & -\mathbf{j} & 0 \end{bmatrix} + x_4 \begin{bmatrix} 0 & \mathbf{k} & 0 & \mathbf{i} \\ \mathbf{k} & 0 & \mathbf{i} & 0 \\ 0 & \mathbf{i} & 0 & \mathbf{k} \\ \mathbf{i} & 0 & \mathbf{k} & 0 \end{bmatrix} \\ &+ x_5 \begin{bmatrix} \mathbf{1} & 0 & \mathbf{j} & 0 \\ 0 & \mathbf{1} & 0 & \mathbf{j} \\ \mathbf{j} & 0 & \mathbf{1} & 0 \\ 0 & \mathbf{j} & 0 & \mathbf{1} \end{bmatrix} + x_6 \begin{bmatrix} -\mathbf{i} & 0 & \mathbf{k} & 0 \\ 0 & -\mathbf{i} & 0 & \mathbf{k} \\ -\mathbf{k} & 0 & \mathbf{i} & 0 \\ 0 & -\mathbf{k} & 0 & \mathbf{i} \end{bmatrix} \\ &= x_1C_1 + x_1C_2 + x_3C_3 + x_4C_4 + x_5C_5 + x_6C_6. \end{aligned}$$

Notice that for each h, since $z_h = x_{2h-1} + x_{2h}\mathbf{i}$, the coefficient matrices C_{2h-1} and C_{2h} have the same supports.

We now review the definition of the Kronecker product:

Definition 10. Let A be an $\ell \times n$ matrix with entries $a_{h,m}$, and B be an $p \times q$ matrix with entries $b_{h,m}$. Then their *Kronecker product*, denoted $A \times B$, is the $\ell p \times nq$ matrix with block entries $\begin{bmatrix} a_{1,1}B & \cdots & a_{1,n}B \\ \vdots & \ddots & \vdots \\ a_{n,1}B & \cdots & a_{n,n}B \end{bmatrix}$. We will also write that $A \times B = [a_{ij}B]$, where it is understood that $1 \le i \le \ell$ and $1 \le j \le n$.

Example 11. Recall from Example 5, we have $W_1 = \begin{bmatrix} w_0 & w_1 \\ -w_1 & w_0 \end{bmatrix}$, where w_0, w_1 are real variables, and let $C_1 = \begin{bmatrix} \mathbf{j} & 0 \\ 0 & \mathbf{j} \end{bmatrix}$ be the first coefficient matrix from Example 8.

Then,

$$\begin{aligned} W_1 \times C_1 &= \begin{bmatrix} w_0C_1 & w_1C_1 \\ -w_1C_1 & w_0C_1 \end{bmatrix} \\ &= \begin{bmatrix} w_0\mathbf{j} & 0 & w_1\mathbf{j} & 0 \\ 0 & w_0\mathbf{j} & 0 & w_1\mathbf{j} \\ -w_1\mathbf{j} & 0 & w_0\mathbf{j} & 0 \\ 0 & -w_1\mathbf{j} & 0 & w_0\mathbf{j} \end{bmatrix}. \end{aligned}$$

Our development thus far allows us to write the following theorem:

Theorem 1. *Let A be a QOD-C with linear processing on complex variables $z_h = x_{2h-1} + x_{2h}\mathbf{i}$, $h = 1, 2, \ldots, u$. Then, writing A in the form*

$$A = \sum_{h=1}^{2u} x_h C_h, \tag{3}$$

where the C_h are the appropriate $k \times n$ coefficient matrices with entries from $\mathcal{Q}$, implies that

$$C_{2h-1}^Q C_{2h-1} = C_{2h}^Q C_{2h} = s_h I_n$$

and

$$C_\ell^Q C_m + C_m^Q C_\ell = 0, \quad \ell \neq m.$$

Proof. As discussed above in Section 1.3, any QOD-C A has entries of the form $\sum_{h=1}^{2u} \alpha_h \mathbf{q_h} x_h$, where the α_h are constants, the $\mathbf{q_h}$ are quaternion elements from $\mathcal{Q}$, and the x_h are real variables; so A can indeed be written as $A = \sum_{h=1}^{2u} x_h C_h$, where C_h are the coefficient matrices. Then, we get:

$$\begin{aligned} A^Q A &= \sum_{h=1}^{2u} x_h C_h^Q (\sum_{h=1}^{2u} x_h C_h) \\ &= \sum_{h=1}^{2u} |x_h|^2 C_h^Q C_h + \sum_{1 \leq \ell < m \leq 2u} x_\ell x_m \left(C_\ell^Q C_m + C_m^Q C_\ell \right). \end{aligned} \tag{4}$$

Also, the orthogonality constraint required by the definition of a QOD-C implies that A satisfies

$$\begin{aligned} A^Q A &= (\sum_{h=1}^{u} s_h |z_h|^2) I_n \\ &= \sum_{h=1}^{u} s_h \left(|x_{2h-1}|^2 + |x_{2h}|^2 \right) I_n. \end{aligned} \tag{5}$$

Now, compare (4) and (5) and notice that the latter equation has no terms of the form $x_\ell x_m$. Therefore, we must have

$$C_{2\ell-1}^Q C_{2\ell-1} = C_{2\ell}^Q C_{2\ell} = s_\ell I_n,$$

and

$$C_\ell^Q C_m + C_m^Q C_\ell = 0, \quad \ell \neq m.$$

Example 12. Given the coefficient matrices in Example 8, wherein all $s_h = 1$, we observe that $C_h^Q C_h = I_2$, $h = 1, \ldots, 4$ and $C_\ell^Q C_m + C_m^Q C_\ell = 0, \quad \ell \neq m$. Similarly, given the coefficient matrices in Example 9, wherein all $s_h = 2$, we observe that $C_h^Q C_h = 2I_4$, $h = 1, \ldots, 6$ and $C_\ell^Q C_m + C_m^Q C_\ell = 0, \quad \ell \neq m$.

We now provide the main result, which provides a construction technique that uses QOD-Cs to build QOD-Rs and certain QOD-Cs that have double the size and the equivalent of one more independent real variable in the alphabet. The resulting designs also have a predictable redundancy that may be important for applications by providing a means for error control.

Theorem 2. (The Amicable-Kronecker Construction) *Suppose there exists a $k \times n$ QOD-C A on u complex variables, with or without linear processing. Then there exists a $2k \times 2n$ QOD-R D on $2u+1$ real variables with each variable appearing once per column and a $2k \times 2n$ QOD-R D on $2u+1$ real variables with all but two variables appearing twice per column.*

Proof. Let A be a QOD-C$(k, n; s_1, \ \dots \ , s_u)$ with linear processing on u complex variables $z_h = x_{2h-1} + x_{2h}\mathbf{i}$ for $h = 1, 2, \dots, u$. Then, we can write $A = \sum_{h=1}^{2u} x_h C_h$, where the C_h are the appropriate $k \times n$ coefficient matrices and for each $h = 1, 2, \dots, u$, C_{2h-1} and C_{2h} must have the same support.

Now, let $w_0, w_1, \dots, w_{2u}$ be real variables. Let $W_1 = \begin{bmatrix} w_0 & w_1 \\ -w_1 & w_0 \end{bmatrix}$, $W_2 = \begin{bmatrix} w_2 & w_2 \\ w_2 & -w_2 \end{bmatrix}$ and for $h > 2$, let $W_h = \begin{bmatrix} w_h & 0 \\ 0 & -w_h \end{bmatrix}$ in the case of $e = 1$ and let $W_h = \begin{bmatrix} w_h & w_h \\ w_h & -w_h \end{bmatrix}$ in the case of $e = 2$. (Recall that the pairwise amicability of these matrices is discussed in Examples 3, 4, and 5.) Then, with $\times$ the Kronecker product, we will show that through cases $e = 1$ and $e = 2$,

$$D = \sum_{h=1}^{2u} W_h \times C_h \tag{6}$$

gives the desired orthogonal designs over the quaternion domain. First, we will show that the columns of D are formally orthogonal. To show this, we will utilize the following facts:

D1 Let $A = [a_{ij}]$ be a real matrix and B be a quaternion matrix. Then, $(A \times B)^Q = A^T \times B^Q$. To see this, note $(A \times B)^Q = ([a_{ij}B])^Q = [a_{ji}B^H] = A^T \times B^H$.

D2 Let A and C be real matrices and B and D be quaternion matrices. Then, when A, B, C, D are suitably sized so that the matrix products AC and BD are well-defined, we have $(A \times B)(C \times D) = (AC \times BD)$. The standard proof via partitioned multiplication for the case of matrices over a field (for example, see [12]) is immediately extendable to the present case as the real elements in C commute with the quaternion elements in B. (Note that this proof does not require A to be real, however that is the case we use below.)

D3 $W_\ell^T W_m = W_m^T W_\ell$ for all $1 \leq \ell < m$, which is true by amicability as shown in Examples 3 - 5.

D4 $C_\ell^Q C_m + C_m^Q C_\ell = 0$ and $C_{2h-1}^Q C_{2h-1} = C_{2h}^Q C_{2h} = s_h I_n$, which hold by Theorem 1.

D5 $W_1^T W_1 = (w_0^2 + w_1^2) I_2$ and $W_h^T W_h = e w_h^2 I_2$, for $h = 2, 3, \ldots, 2u$, as readily confirmed by inspection.

We are now prepared to consider $D^Q D$:

$$
\begin{aligned}
D^Q D &= (W_1 \times C_1 + \cdots + W_{2u} \times C_{2u})^Q (W_1 \times C_1 + \cdots + W_{2u} \times C_{2u}) \\
&= (W_1^T \times C_1^Q + \cdots + W_{2u}^T \times C_{2u}^Q)(W_1 \times C_1 + \cdots + W_{2u} \times C_{2u}) \quad \text{(by } \mathbf{D1}\text{)} \\
&= (W_1^T \times C_1^Q)(W_1 \times C_1 + \cdots + W_{2u} \times C_{2u}) + \\
&\quad (W_2^T \times C_2^Q)(W_1 \times C_1 + \cdots + W_{2u} \times C_{2u}) + \cdots + \\
&\quad (W_{2u}^T \times C_{2u}^Q)(W_1 \times C_1 + \cdots + W_{2u} \times C_{2u}) \qquad \text{(by distributing)} \\
&= (W_1^T W_1 \times C_1^Q C_1) + \cdots + (W_1^T W_{2u} \times C_1^Q C_{2u}) + \\
&\quad (W_2^T W_1 \times C_2^Q C_1) + \cdots + (W_2^T W_{2u} \times C_2^Q C_{2u}) + \cdots + \\
&\quad (W_{2u}^T W_1 \times C_{2u}^Q C_1) + \cdots + (W_{2u}^T W_{2u} \times C_{2u}^Q C_{2u}) \qquad \text{(by } \mathbf{D2}\text{)} \\
&= \sum_{h=1}^{2u} W_h^T W_h \times C_h^Q C_h + \sum_{1 \le \ell < m \le 2u} \left(W_\ell^T W_m \times C_\ell^Q C_m + W_m^T W_\ell \times C_m^Q C_\ell \right) \\
&\qquad \text{(collecting terms)} \\
&= \sum_{h=1}^{2u} W_h^T W_h \times C_h^Q C_h + \sum_{1 \le \ell < m \le 2u} W_\ell^T W_m \times \left(C_\ell^Q C_m + C_m^Q C_\ell \right) \\
&\qquad \text{(by } \mathbf{D2} \text{ and } \mathbf{D3}\text{)} \\
&= \sum_{h=1}^{2u} W_h^T W_h \times C_h^Q C_h \qquad \text{(by } \mathbf{D4}\text{)} \\
&= (w_0^2 + w_1^2) I_2 \times s_1 I_n + \Big(\sum_{h=2}^{2u} e w_h^2 I_2\Big) \times s_h I_n \quad \text{(by } \mathbf{D4} \text{ and } \mathbf{D5}\text{)} \\
&= (s_1 w_0^2 + s_1 w_1^2) I_{2n} + \sum_{h=2}^{2u} s_h e w_h^2 I_{2n} \qquad \text{(by Kronecker product)} \\
&= (s_1 w_0^2 + s_1 w_1^2 + \sum_{h=2}^{2u} s_h e w_h^2) I_{2n}.
\end{aligned}
$$

This shows that $D^Q D$ is a diagonal matrix, so its columns are formally orthogonal as required.

Now we will examine the entries of D. In both cases of $e = 1$ and $e = 2$, by the definition of W_h for all h and since C_{2h-1} and C_{2h} have the same support for all h, the entries of D will contain quaternion combinations of w_0 with w_2, quaternion combinations of w_1 and w_2, or quaternion combinations of w_{2h-1} with w_{2h} for $h =$

$2, 3, \ldots, u$. As such, the resulting design D will be a QOD-R on $2u+1$ real variables $w_0, w_1, \ldots w_{2u}$ with linear processing. It is also possible to view D as a QOD-C on $u+1$ complex variables $v_1 = w_0 + w_2\mathbf{i}$, $v_2 = w_1 + w_2\mathbf{i}$, $v_3 = w_3 + w_4\mathbf{i}$, $v_4 = w_5 + w_6\mathbf{i}$, $\ldots$, $v_{u+1} = w_{2u-1} + w_{2u}\mathbf{i}$, however v_1 and v_2 are not independent complex variables: they have different real terms but the same imaginary terms.

The cases of $e = 1$ and $e = 2$ differ in how many times each real variable component is repeated in each column. If we begin with a design A with no linear processing and $s_h = 1$ for all h, then $e = 1$ will give a QOD-R of type $(2k, 2n; 1, 1, 2, 1, 1, \ldots, 1)$ and $e = 2$ will give a QOD-R of type $(2k, 2n; 1, 1, 2, 2, \ldots, 2)$. Changing a particular s_h will multiply the type of the corresponding entries appropriately.

Thus, given an initial QOD-C A (readily available for any number of columns using existing constructions [3, 16]), we have shown a method for applying the Kronecker product with a simple family of real amicable orthogonal designs to build a QOD-R (or certain QOD-C's) with double the size and the equivalent of one more independent real variable, which is important for increasing the amount of information that can be conveyed by a design. The construction also allows us to provide redundancy in the real variables in each column, a potential benefit for coding applications.

□

To summarize the results implied by Theorem 2, we provide Table I which shows the parameters of input QOD-Cs with up to 10 columns and the resulting parameters of the output QOD-Rs with up to 20 columns. We follow the convention that optimal parameters of an orthogonal matrix equate to the matrix having maximum rate (i.e., maximum ratio of number of variables to number of rows) and minimum decoding delay (i.e., minimum number of rows for a matrix with a given rate and given number of columns) [18]. Thus, it is natural to use input QOD-Cs that are built (via Theorem 4 of [16] or Theorem 1 of [3] for the case of an even number of columns) from and inherent the parameters of complex orthogonal designs (CODs) of maximum rate and minimum decoding delay: the optimal parameters are known for CODs for every number of columns; such CODs can be easily constructed using existing techniques; and constructions exist that use CODs to build QOD-Cs with the same parameters. Thus, we recall that for a COD with $n = 2m-1$ or $n = 2m$ columns, the maximum rate is $\frac{m+1}{2m}$ [14]. Furthermore, for such a COD of maximum rate, the minimum decoding delay is $\binom{2m}{m-1}$ when n is congruent to 0, 1 or 3 modulo 4 [2] and $2\binom{2m}{m-1}$ when n is congruent to 2 modulo 4 [1]. Finally, we can interpret Table I as follows: the first column indicates the dimensions of the input QOD-C (which inherits its parameters from a maximum rate, minimum decoding delay COD via Theorem 4 of [16] or Theorem 1 of [3]); the second column indicates the number of complex variables in this input QOD-C; the third column indicates the dimensions of the output QOD-R obtained via Theorem 2; and the fourth column indicates the number of real variables in the resulting QOD-R.

Although it is possible to apply Theorem 2 multiple times to achieve varied results, Table I is a summary of the QOD-Rs that arise from one simple application

Table 1: Table I. Parameters of input QOD-C and output QOD-R via Theorem 2

Size input QOD-C	No. complex var. input	Size output QOD-R	No. real var. output
$k \times n$	u	$2k \times 2n$	$2u+1$
2×2	2	4×4	5
4×3	3	8×6	7
4×4	3	8×8	7
15×5	10	30×10	21
30×6	20	60×12	41
56×7	35	112×14	71
56×8	35	112×16	71
210×9	126	420×18	253
420×10	252	840×20	505

of the theorem.

Example 13. Recall from Example 8 the QOD-C A of type $(2,2;1,1)$:

$$\begin{aligned} A &= \begin{bmatrix} z_1\mathbf{j} & z_2 \\ z_2 & z_1\mathbf{j} \end{bmatrix} \\ &= x_1 \begin{bmatrix} \mathbf{j} & 0 \\ 0 & \mathbf{j} \end{bmatrix} + x_2 \begin{bmatrix} \mathbf{k} & 0 \\ 0 & \mathbf{k} \end{bmatrix} + x_3 \begin{bmatrix} 0 & \mathbf{1} \\ \mathbf{1} & 0 \end{bmatrix} + x_4 \begin{bmatrix} 0 & \mathbf{i} \\ \mathbf{i} & 0 \end{bmatrix} \\ &= x_1C_1 + x_2C_2 + x_3C_3 + x_4C_4. \end{aligned}$$

Note that A is a 2×2 QOD-C on 2 complex variables, matching the parameters of an optimal COD. First, we will apply Theorem 2, the Amicable-Kronecker Construction, to A with $e=1$ to obtain a 4×4 QOD-R $D_{(e=1)}$ of type $(4,4;1,1,2,1,1)$. According to (6) in the proof of Theorem 2, we consider $D_{(e=1)} = (W_1 \times C_1) + (W_2 \times C_2) + (W_3 \times C_3) + (W_4 \times C_4)$, where $W_1 = \begin{bmatrix} w_0 & w_1 \\ -w_1 & w_0 \end{bmatrix}$, $W_2 = \begin{bmatrix} w_2 & w_2 \\ w_2 & -w_2 \end{bmatrix}$, and $W_m = \begin{bmatrix} w_m & 0 \\ 0 & -w_m \end{bmatrix}$, for $m = 3, 4$. Note that we have already computed $W_1 \times C_1$ in Example 11, and we have already seen in Examples 3 and 5 that the W_m, $m = 1,\ 2,\ 3,\ 4$ are pairwise amicable.

We have

$$\begin{aligned} D_{(e=1)} &= (W_1 \times C_1) + (W_2 \times C_2) + (W_3 \times C_3) + (W_4 \times C_4) \\ &= \begin{bmatrix} w_0C_1 & w_1C_1 \\ -w_1C_1 & w_0C_1 \end{bmatrix} + \begin{bmatrix} w_2C_2 & w_2C_2 \\ w_2C_2 & -w_2C_2 \end{bmatrix} + \begin{bmatrix} w_3C_3 & (0)C_3 \\ (0)C_3 & -w_3C_3 \end{bmatrix} + \begin{bmatrix} w_4C_4 & (0)C_4 \\ (0)C_4 & -w_4C_4 \end{bmatrix} \end{aligned}$$

$$= \begin{bmatrix} w_0\mathbf{j} & 0 & w_1\mathbf{j} & 0 \\ 0 & w_0\mathbf{j} & 0 & w_1\mathbf{j} \\ -w_1\mathbf{j} & 0 & w_0\mathbf{j} & 0 \\ 0 & -w_1\mathbf{j} & 0 & w_0\mathbf{j} \end{bmatrix} + \begin{bmatrix} w_2\mathbf{k} & 0 & w_2\mathbf{k} & 0 \\ 0 & w_2\mathbf{k} & 0 & w_2\mathbf{k} \\ w_2\mathbf{k} & 0 & -w_2\mathbf{k} & 0 \\ 0 & w_2\mathbf{k} & 0 & -w_2\mathbf{k} \end{bmatrix}$$
$$+ \begin{bmatrix} 0 & w_3 & 0 & 0 \\ w_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & -w_3 \\ 0 & 0 & -w_3 & 0 \end{bmatrix} + \begin{bmatrix} 0 & w_4\mathbf{i} & 0 & 0 \\ w_4\mathbf{i} & 0 & 0 & 0 \\ 0 & 0 & 0 & -w_4\mathbf{i} \\ 0 & 0 & -w_4\mathbf{i} & 0 \end{bmatrix}$$
$$= \begin{bmatrix} w_0\mathbf{j}+w_2\mathbf{k} & w_3+w_4\mathbf{i} & w_1\mathbf{j}+w_2\mathbf{k} & 0 \\ w_3+w_4\mathbf{i} & w_0\mathbf{j}+w_2\mathbf{k} & 0 & w_1\mathbf{j}+w_2\mathbf{k} \\ -w_1\mathbf{j}+w_2\mathbf{k} & 0 & w_0\mathbf{j}-w_2\mathbf{k} & -w_3-w_4\mathbf{i} \\ 0 & -w_1\mathbf{j}+w_2\mathbf{k} & -w_3-w_4\mathbf{i} & w_0\mathbf{j}-w_2\mathbf{k} \end{bmatrix}.$$

This shows that $D_{(e=1)}$ is naturally viewed as a QOD-R with linear processing on real variables $w_0, \ldots, w_5$, and $D_{(e=1)}$ is of type QOD-R$(4,4;1,1,2,1,1)$. The novelty and benefit of this construction is that we have a 4×4 orthogonal structure that uses five independent real variables. In contrast, a traditional 4×4 real orthogonal design could incorporate at most four real variables. A traditional 4×4 complex orthogonal design could incorporate up to three complex variables, effectively six real variables, however such designs do not have any quaternion coefficients which may be useful for modeling polarization in wireless systems.

Now, writing $v_1 = w_0 + w_2\mathbf{i}$, $v_2 = w_1 + w_2\mathbf{i}$ and $v_3 = w_3 + w_4\mathbf{i}$ gives

$$D'_{(e=1)} = \begin{bmatrix} v_1\mathbf{j} & v_3 & v_2\mathbf{j} & 0 \\ v_3 & v_1\mathbf{j} & 0 & v_2\mathbf{j} \\ -v_2^*\mathbf{j} & 0 & v_1^*\mathbf{j} & -v_3 \\ 0 & -v_2^*\mathbf{j} & -v_3 & v_1^*\mathbf{j} \end{bmatrix}.$$

So, $D'_{(e=1)}$ can also be viewed as a QOD-C on the complex variables v_1, v_2, v_3 as expected by the proof of Theorem 2, however v_1 and v_2 are not independent but rather have the same imaginary component. So, by applying Theorem 2 with $e = 1$ to a 2×2 QOD-C A of type $(2,2;1,1)$, we obtained $D_{(e=1)}$ a QOD-R(4, 4; 1, 1, 2, 1, 1) and $D'_{(e=1)}$, a QOD-C(4,4; 1,1,1) whose first two variables are dependent.

Now we will apply Theorem 2 to A with $e = 2$ to obtain a 4×4 QOD-R $D_{(e=2)}$ of type $(4,4;1,1,2,2,2)$ with linear processing. According to (6) in the proof of Theorem 2, we consider $D_{(e=2)} = (W_1 \times C_1) + (W_2 \times C_2) + (W_3 \times C_3) + (W_4 \times C_4)$, where $W_1 = \begin{bmatrix} w_0 & w_1 \\ -w_1 & w_0 \end{bmatrix}$, and $W_m = \begin{bmatrix} w_m & w_m \\ w_m & -w_m \end{bmatrix}$, for $m = 2,3,4$. We have already seen in Examples 4 and 5 that the W_m, $m = 1,\ 2,\ 3,\ 4$ are pairwise amicable.

We have

$$\begin{aligned} D_{(e=2)} &= (W_1 \times C_1) + (W_2 \times C_2) + (W_3 \times C_3) + (W_4 \times C_4) \\ &= \begin{bmatrix} w_0C_1 & w_1C_1 \\ -w_1C_1 & w_0C_1 \end{bmatrix} + \begin{bmatrix} w_2C_2 & w_2C_2 \\ w_2C_2 & -w_2C_2 \end{bmatrix} + \begin{bmatrix} w_3C_3 & w_3C_3 \\ w_3C_3 & -w_3C_3 \end{bmatrix} + \begin{bmatrix} w_4C_4 & w_4C_4 \\ w_4C_4 & -w_4C_4 \end{bmatrix} \end{aligned}$$

$$= \begin{bmatrix} w_0\mathbf{j} & 0 & w_1\mathbf{j} & 0 \\ 0 & w_0\mathbf{j} & 0 & w_1\mathbf{j} \\ -w_1\mathbf{j} & 0 & w_0\mathbf{j} & 0 \\ 0 & -w_1\mathbf{j} & 0 & w_0\mathbf{j} \end{bmatrix} + \begin{bmatrix} w_2\mathbf{k} & 0 & w_2\mathbf{k} & 0 \\ 0 & w_2\mathbf{k} & 0 & w_2\mathbf{k} \\ w_2\mathbf{k} & 0 & -w_2\mathbf{k} & 0 \\ 0 & w_2\mathbf{k} & 0 & -w_2\mathbf{k} \end{bmatrix}$$

$$+ \begin{bmatrix} 0 & w_3 & 0 & w_3 \\ w_3 & 0 & w_3 & 0 \\ 0 & w_3 & 0 & -w_3 \\ w_3 & 0 & -w_3 & 0 \end{bmatrix} + \begin{bmatrix} 0 & w_4\mathbf{i} & 0 & w_4\mathbf{i} \\ w_4\mathbf{i} & 0 & w_4\mathbf{i} & 0 \\ 0 & w_4\mathbf{i} & 0 & -w_4\mathbf{i} \\ w_4\mathbf{i} & 0 & -w_4\mathbf{i} & 0 \end{bmatrix}$$

$$= \begin{bmatrix} w_0\mathbf{j} + w_2\mathbf{k} & w_3 + w_4\mathbf{i} & w_1\mathbf{j} + w_2\mathbf{k} & w_3 + w_4\mathbf{i} \\ w_3 + w_4\mathbf{i} & w_0\mathbf{j} + w_2\mathbf{k} & w_3 + w_4\mathbf{i} & w_1\mathbf{j} + w_2\mathbf{k} \\ -w_1\mathbf{j} + w_2\mathbf{k} & w_3 + w_4\mathbf{i} & w_0\mathbf{j} - w_2\mathbf{k} & -w_3 - w_4\mathbf{i} \\ w_3 + w_4\mathbf{i} & -w_1\mathbf{j} + w_2\mathbf{k} & -w_3 - w_4\mathbf{i} & w_0\mathbf{j} - w_2\mathbf{k} \end{bmatrix}.$$

This shows that $D_{(e=2)}$ is naturally viewed as a QOD-R with linear processing on real variables $w_0, \ldots, w_5$, and $D_{(e=2)}$ is of type QOD-R$(4, 4; 1, 1, 2, 2, 2)$. This QOD-R has the same benefits of $D_{(e=1)}$, with the additional benefit of having no zero entries, which may be preferred in applications [17, 19, 23]. Also, all but two of the real variables are repeated in each column, which may be an exploitable redundancy pattern.

Using $D_{(e=2)}$, we can write $v_1 = w_0 + w_2\mathbf{i}$, $v_2 = w_1 + w_2\mathbf{i}$ and $v_3 = w_3 + w_4\mathbf{i}$ to give

$$D'_{(e=2)} = \begin{bmatrix} v_1\mathbf{j} & v_3 & v_2\mathbf{j} & v_3 \\ v_3 & v_1\mathbf{j} & v_3 & v_2\mathbf{j} \\ -v_2^*\mathbf{j} & v_3 & v_1^*\mathbf{j} & -v_3 \\ v_3 & -v_2^*\mathbf{j} & -v_3 & v_1^*\mathbf{j} \end{bmatrix},$$

which is an QOD-C on the complex variables v_1, v_2, v_3, however v_1 and v_2 are not independent. So, by applying Theorem 2 to the 2×2 QOD-C A of type $(2, 2; 1, 1)$, we obtained $D_{(e=2)}$ of type QOD-R$(4, 4; 1, 1, 2, 2, 2)$ and $D'_{(e=2)}$ of type QOD-C$(4, 4; 1, 1, 2)$, wherein the first two complex variables are not independent.

In summary, Theorem 2 can be used to transform a QOD-C A of size $k \times n$ on u complex variables into a QOD-R D of size $2k \times 2n$ on $2u + 1$ real variables. It is then possible to pair together the real variables in D to form complex variables and then view the matrix as a QOD-C D' over $u + 1$ complex variables, two of which are dependent. The benefit of viewing the resulting matrix as a QOD-C (despite the complex variables not being fully independent) is that we can then re-apply Theorem 2 (possibly varying the order in which we use cases $e = 1$ and $e = 2$) to obtain QOD-Rs of varying sizes and types.

As an important final note, consider $D_{(e=2)}$ from Example 13. We may add any two columns of $D_{(e=2)}$ together, without impacting the orthogonality of the columns. Therefore, we consider the 4×2 matrix D'' whose two columns are, respectively, the sum of the first and second columns of $D_{(e=2)}$ and the sum of the third and fourth columns of $D_{(e=2)}$:

$$D'' = \begin{bmatrix} w_3 + w_4\mathbf{i} + w_0\mathbf{j} + w_2\mathbf{k} & w_3 + w_4\mathbf{i} + w_1\mathbf{j} + w_2\mathbf{k} \\ w_3 + w_4\mathbf{i} + w_0\mathbf{j} + w_2\mathbf{k} & w_3 + w_4\mathbf{i} + w_1\mathbf{j} + w_2\mathbf{k} \\ w_3 + w_4\mathbf{i} - w_1\mathbf{j} + w_2\mathbf{k} & -w_3 - w_4\mathbf{i} + w_0\mathbf{j} - w_2\mathbf{k} \\ w_3 + w_4\mathbf{i} - w_1\mathbf{j} + w_2\mathbf{k} & -w_3 - w_4\mathbf{i} + w_0\mathbf{j} - w_2\mathbf{k} \end{bmatrix}.$$

Notice that the columns of D'' are orthogonal and $(D'')^Q(D'') = 4\sum_{h=1}^{4} w_h I_2$. Also, notice that the entries of D'' are full quaternion variables, or in other words, the entries are quaternion variables that are non-zero in every component of the quaternion variable. Therefore, we say that D'' is QOD-Q. In this example D'', the quaternion variables in each entry are not independent, however they include five independent real variables and this redundancy may be useful when implemented in a wireless system by providing some error control. QOD-Q's were initially more challenging to construct than QOD-R's and QOD-C's, due to the difficulties introduced when handling entries that are full quaternion variables [16], however a later construction was introduced that provides QOD-Qs for any number of columns (see Theorem 4 in [3]). An advantage of this example D'' is that it has no zero entries, which is a desirable property for applications [17, 19, 23]. Only some of the sporadic 2×2 examples demonstrated in [16] also have this property, where the two quaternion variables have components that satisfy a variety of dependencies. The more general construction in [3] necessarily gives QOD-Qs with zero entries in every column for any number of columns greater than 2.

3 Conclusions

Quaternion orthogonal designs of various types (QOD-R, QOD-C, and QOD-Q) have been introduced for possible use in wireless communications systems [16]. Such QODs are interesting mathematical objects, but due to system requirements, only certain QODs will be useful in applications [20]. This increases the need for additional construction techniques to widen the search for applicable QODs. In this paper, we provided a new construction technique using real amicable orthogonal designs and the Kronecker product. We were able to use small QOD-C's to build larger QOD-R's with the equivalent of one additional independent real variable. In some cases, the resulting designs have the desirable property of no zero entries and all but two real variables repeating exactly twice per column. This construction also allowed us to build QOD-C's and QOD-Q's whose variables include certain dependencies. It is an interesting open problem to determine if using alternative families of real amicable orthogonal designs can significantly improve the QODs that result from the Amicable-Kronecker Construction.

Acknowledgements

The authors thank anonymous reviewers for their careful reading of this work.

References

[1] S. Adams, N. Karst and J. Pollack, The final case of the decoding delay problem for maximum rate complex orthogonal designs, *IEEE Trans. Inform. Theory* 56(1) (2010), 103–112.

[2] S. Adams, N. Karst and J. Pollack, The minimum decoding delay of maximum rate complex ortogonal space-time block codes, *IEEE Trans. Inform. Theory* 53(8) (2007), 2677–2684.

[3] S.S. Adams, J. Seberry, N. Karst, J. Pollack and T. Wysocki, Quaternion orthogonal designs from complex companion designs, *Linear Algebra Applic.* 428(4) (2008), 1056–1071.

[4] S.M. Alamouti, A simple transmit diversity technique for wireless communications. *IEEE J. Selected Areas Commun.* 16(8) (1998), 1451–1458.

[5] A.R. Calderbank, S.Das, N. Al-Dhahir and S.N. Diggavi, Construction and analysis of a new quaternionic space-time code for 4 transmit antennas, *Commun. Inf. Systems* 5(1) (2005), 1–26.

[6] B.S. Collins, Polarization-diversity antennas for compact base stations, *Microwave J.* 43(1) (2000), 76–88.

[7] B.S. Collins, The effect of imperfect antenna cross-polar performance on the diversity gain of a polarization-diversity system, *Microwave J.* 43(4) (2000), 84–94.

[8] C.B. Dietrich, K. Dietze, J.R. Nealy and W.L. Stulzman, Spatial polarization and pattern diversity for wireless handheld terminals, *IEEE Trans. Antennas Propagat.* 49 (2001), 1271–1281.

[9] K. Finlayson, J. Seberry, T. Wysocki and T. Xia, Orthogonal designs with quaternion elements. *Proc. 8th Int. Symposium on Communication Theory and Applications*, Ambleside, UK, 2005, 270–272.

[10] A.V. Geramita, and J. Seberry, *Orthogonal Designs: Quadratic Forms and Hadamard Matrices*, Marcel Dekker, New York-Basel, 1979.

[11] A.V. Geramita, J.M. Geramita and J. Seberry Wallis, Orthogonal designs, *Linear and Multilinear Algebra* 3 (1976), 281–306.

[12] R. Horn and C. Johnson, *Topics in Matrix Analysis*, Cambridge University Press, 1985.

[13] O.M. Isaeva and V.A. Sarytchev, Quaternion presentations polarization state, *Proc. 2nd IEEE Topical Symposium of Combined Optical-Microwave Earth and Atmosphere Sensing*, Atlanta, GA, USA (1995), 195–196.

[14] X.-B. Liang, Orthogonal designs with maximal rates, *IEEE Trans. Inform. Theory*, (Special issue on space-time transmission, reception, coding and signal processing), 49(10) (2003), 2468–2503.

[15] R.U. Nabar, H.Bölcskei, V. Ereg, D. Gesbert and A.J. Paulraj, Performance of multiantenna signalling techniques in the presence of polarization diversity, *IEEE Trans. Signal Processing* 50 (2002), 2553–2562.

[16] J. Seberry, K. Finlayson, S.S. Adams, T. Wysocki, T. Xia and B. Wysocki, The theory of quaternion orthogonal designs, *IEEE Trans. Signal Processing* 56(1) (2008), 256–265.

[17] J. Seberry, S.A. Spence and T. Wysocki, A construction technique for generalized complex orthogonal designs and applications to wireless communications, *Linear Algebra Applic.* 405 (2005), 163–176.

[18] V. Tarokh, H. Jafarkhani and A.R. Calderbank, Space-time block codes from orthogonal designs, *IEEE Trans. Inform. Theory* 45(5) (1999), 1456–1467.

[19] L. Tran, T. Wysocki, J. Seberry, A. Mertins and S. Spence, Generalized Williamson and Wallis-Whiteman constructions for improved square order-8 CO STBCs, *Proc. IEEE 16th Int. Symposium on Personal, Indoor and Mobile Radio Communications* 2 (2005), 1155–1159.

[20] T. Wysocki, B. Wysocki and S.S. Adams, Correction to 'The Theory of Quaternion Orthogonal Designs', *IEEE Trans. Signal Processing* 57(8) (2009), 3298–3298.

[21] C. Yuen, Y.L. Guan and T.T. Tjhung, Algebraic Relationship between Quasi-Orthgonal STBC with Minimum Decoding Complexity and Amicable Orthogonal Design, *Proc. IEEE Int. Conf. Communications* (2006), 4882–4887.

[22] C. Yuen, Y.L. Guan and T.T. Tjhung, Amicable Complex Orthogonal Designs, *Australas. J. Combin.* 44 (2009), 111–121.

[23] C. Yuen, Y.L. Guan and T.T. Tjhung, Power-balanced orthogonal space-time block code, *IEEE Trans. Vehicular Technology* 57(5) (2008), 3304–3309.

[24] Z. Zhao, S. Stapleton and J.K. Cavers, Analysis of polarization diversity scheme with channel codes, *Proc. IEEE Vehicular Tech. Conference* (1999), 1337–1381.

(Received 12 Jan 2011; revised 21 Apr 2011)