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New classes of orthogonal designs constructed from complementary sequences with given spread

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Abstract

In this paper we present infinite families of new orthogonal designs, based on some weighing matrices of order $2n$, weight $2n-k$ and spread σ, constructed from two circulants and directed sequences.

1 Introduction

A weighing matrix $W = W(n, w)$ is a square matrix with entries $0, \pm 1$ having w non-zero entries per row and column and inner product of distinct rows equal to zero. Therefore W satisfies $WW^T = wI_n$. The number w is called the weight of W. Weighing matrices have been studied extensively; see [1], [2] and [5] and references therein, for detailed information on known and unknown weighing matrices.

A well-known necessary condition for the existence of $W(2n, w)$ matrices states that if there exists a $W(2n, w)$ matrix with n odd, then $w < 2n$ and w is the sum of two squares. In this paper we are focusing on $W(2n, w)$ constructed from two circulants. The two circulants construction for weighing matrices is described in the theorem below, taken from [3].

Theorem 1 *If there exist two circulant matrices A_1, A_2 of order n, with $0, \pm 1$ entries, satisfying $A_1A_1^T + A_2A_2^T = wI_n$, then there exists a $W(2n, w)$, given as*

$$W(2n,w) = \begin{pmatrix} A_1 & A_2 \\ -A_2^T & A_1^T \end{pmatrix} \quad or \quad W(2n,w) = \begin{pmatrix} A_1 & A_2R \\ -A_2R & A_1 \end{pmatrix}$$

where R is the square matrix of order n with $r_{ij} = 1$ if $i + j - 1 = n$ and 0 otherwise.

The orthogonal design constructions of this paper are based on formulations of a theorem of Goethals and Seidel which may be used in the following form:

Theorem 2 [3, Theorem 4.49] *If there exist four circulant matrices A_1, A_2, A_3, A_4 of order n satisfying*

$$\sum_{i=1}^{4} A_iA_i^T = fI$$

where f is the quadratic form $\sum_{j=1}^{u} s_jx_j^2$, then there is an orthogonal design $OD(4n; s_1, s_2, \dots, s_u)$.

2 The spread of two sequences with PAF zero

In this section we introduce the concept of the spread of sequences.

Definition 1 *Let $A = [a_1, a_2, \dots, a_n]$ be a sequence of length n. The* periodic autocorrelation function, PAF, *$P_A(s)$ is defined as:*

$$P_A(s) = \sum_{i=1}^{n} a_ia_{i+s} \; , s = 0, 1, \dots, n-1$$

where we consider i+s modulo n.

Definition 2 *Let $A = [a_1, a_2, \dots, a_n]$ be a sequence of length n. The* non-periodic autocorrelation function, NPAF, *$N_A(s)$ is defined as:*

$$N_A(s) = \sum_{i=1}^{n-s} a_ia_{i+s} \; , s = 0, 1, \dots, n-1.$$

Definition 3 *Two sequences, $A = [a_1, \cdots, a_n]$ and $B = [b_1, \cdots, b_n]$, of length n are said to have* zero PAF *(respectively* zero NPAF*), if $P_A(s) + P_B(s) = 0$ (respectively if $N_A(s) + N_B(s) = 0$) where we consider $i+s$ modulo n for $s = 1, \ldots, n-1$.*

Definition 4 *Two sequences, $A = [a_1, \cdots, a_n]$ and $B = [b_1, \cdots, b_n]$, of length n and elements from $\{-1, +1\}$ are called* Golay sequences *if they have zero NPAF, i.e. if $N_A(s) + N_B(s) = 0$ for $s = 1, \ldots, n-1$.*

Definition 5 *A sequence $A = [a_1, \cdots, a_n]$ of length n is said to have* **spread** *$s = s(A)$, if the largest block of consecutive zeros is of length s.*

Note that in a sequence $A = [a_1, \cdots, a_n]$ with spread s there may be more than s zero elements in total.

Example 1 *The following sequences of length $n = 7$, have spread $s = 3$:*

- $[0, 0, 0, \underbrace{a_4}_{\neq 0}, a_5, a_6 \underbrace{a_7}_{\neq 0}]$ *where a_5, a_6 may be zero*
- $[\underbrace{a_1}_{\neq 0}, 0, 0, 0, \underbrace{a_5}_{\neq 0}, a_6, a_7]$ *where a_6, a_7 may be zero*

Definition 6 *Two sequences with PAF zero, $A = [a_1, \cdots, a_n]$ and $B = [b_1, \cdots, b_n]$, of length n are said to have* spread *$s = s(A, B)$, if $s = \min(s(A), s(B))$.*

Two sequences of length n that have spread s, each contain at least s consecutive zero elements.

Example 2 *The following pairs of sequences of length $n = 7$, have spread $s = 3$:*

- $[0, 0, 0, \underbrace{a_4,}_{\neq 0} a_5, a_6, \underbrace{a_7}_{\neq 0}]$ *and* $[0, 0, 0, 0, \underbrace{b_5,}_{\neq 0} b_6, \underbrace{b_7}_{\neq 0}]$
- $[\underbrace{a_1,}_{\neq 0} 0, 0, 0, \underbrace{a_5,}_{\neq 0} a_6, a_7]$ *and* $[\underbrace{b_1,}_{\neq 0} 0, 0, 0, 0, \underbrace{b_6,}_{\neq 0} b_7]$

Definition 7 *Two sequences with PAF zero, $A = [a_1, \cdots, a_n]$ and $B = [b_1, \cdots, b_n]$, of length n and spread s are called* spread-normalized *if they are each shifted cyclically so that the longest sets of consecutive zeros appear in positions $1, 2, \ldots, s(A)$ and $1, 2, \ldots, s(B)$ in A and B respectively.*

Example 3 *The following pairs of sequences of length $n = 7$, have spread $s = 3$ and are spread-normalized:*

- $[0, 0, 0, \underbrace{a_4,}_{\neq 0} a_5, a_6, \underbrace{a_7}_{\neq 0}]$ *and* $[0, 0, 0, 0, \underbrace{b_5,}_{\neq 0} b_6, \underbrace{b_7}_{\neq 0}]$
- $[0, 0, 0, 0, \underbrace{a_5,}_{\neq 0} a_6, \underbrace{a_7}_{\neq 0}]$ *and* $[0, 0, 0, \underbrace{b_4,}_{\neq 0} b_5, b_6, \underbrace{b_7}_{\neq 0}]$

We also require the following definition taken from [5]:

Definition 8 *Two sequences of length n, with zero PAF or zero NPAF, are said to be of type (u, v) if the sequences are composed of two variables, say a and b, so that a and $-a$ occur a total of u times and b and $-b$ occur a total of v times.*

Note that two sequences of type (u, v) can be used as the first rows of two circulant matrices in Theorem 1 to obtain an $OD(2n; u, v)$.

3 New orthogonal designs from $W(2n, 2n-k)$

We now show how to use sequences with zero PAF and spread s to construct new orthogonal designs. The sequences with zero PAF come from $W(2n, 2n-k)$ weighing matrices constructed from two circulants.

Theorem 3 *Suppose there exists a weighing matrix $W(2n, 2n-k)$ constructed from two circulants, whose first rows have spread σ. Suppose there exist two sequences of length σ, with NPAF zero and type (u, v). Then there exists an $OD(4n; 2u, 2v, 4n-2k)$.*

Proof
Let C and D be the first rows of the two circulants that make up the weighing matrix $W(2n, 2n-k)$. We spread-normalize C and D and call the resulting $\{0, \pm 1\}$ sequences A and B respectively. We multiply A and B by the commuting variable α to obtain:

$$\begin{array}{rl} A = & [\underbrace{0, \cdots, 0,}_{\sigma \text{ zeros}} \ a_{\sigma+1}, \ \cdots, \ a_n] \\ B = & [\underbrace{0, \cdots, 0,}_{\sigma \text{ zeros}} \ b_{\sigma+1}, \ \cdots, \ b_n] \end{array} \qquad (1)$$

where $a_k, b_k \in \{0, \pm\alpha\}, k = \sigma+1, \ldots, n$ and either $a_{\sigma+1} \neq 0$ or $b_{\sigma+1} \neq 0$.
Now denote the two sequences of length σ, with NPAF zero and type (u, v) by $P' = [p_1, \cdots, p_\sigma]$ and $Q' = [q_1, \cdots, q_\sigma]$. Then the $OD(4n; 2u, 2v, 4n-2k)$ can be constructed by forming the four circulant matrices with the given first rows P, Q, R, S below, which are then used in the Goethals-Seidel array.

$$\begin{array}{rlllllll} P = [& p_1, & p_2, & \cdots, & p_\sigma, & a_{\sigma+1}, & \cdots, & a_n] \\ Q = [& -p_1, & -p_2, & \cdots, & -p_\sigma, & a_{\sigma+1}, & \cdots, & a_n] \\ R = [& q_1, & q_2, & \cdots, & q_\sigma, & b_{\sigma+1}, & \cdots, & b_n] \\ S = [& -q_1, & -q_2, & \cdots, & -q_\sigma, & b_{\sigma+1}, & \cdots, & b_n] \end{array} \qquad (2)$$

The sequences P, Q, R, S have PAF zero. This is because any products that arise in the PAF of P from elements of P' with the elements of the sequence $[a_{\sigma+1}, \cdots, a_n]$ are negated in the PAF of Q by the product of the elements of $-P'$ with the elements of the sequence $[a_{\sigma+1}, \cdots, a_n]$. We have the same for Q' and the sequence $[b_{\sigma+1}, \cdots, b_n]$ in R and S. The sum of the products from elements of the sequence $[a_{\sigma+1}, \cdots, a_n]$ with the products from elements of the sequence $[b_{\sigma+1}, \cdots, b_n]$ is equal to zero in the

PAF of P, Q, R, S, since the sequences A and B have PAF zero. We have the same for the sum of the products from elements of P' with the products from elements of Q' in the PAF of P, Q, R, S, since these sequences have NPAF zero. This gives the required $OD(4n; 2u, 2v, 4n-2k)$.

□

Theorem 3 has a corollary pertaining to directed sequences. We will say that some sequences of variables are *directed* if the sequences have zero autocorrelation function independently from the properties of the variables, i.e. they do not rely on commutativity to ensure zero autocorrelation (PAF or NPAF). For example $X = [a, b]$ and $Y = [a, -b]$ are two directed sequences ($N_X(1) + N_Y(1) = ab - ab = 0$ without relying on commutativity) while $Z = [a, b]$ and $W = [b, -a]$ are not directed ($N_Z(1) + N_W(1) = ab - ba = 0$ if and only if a, b are commuting variables). Directed sequences were introduced in [5] and have been used extensively so far to construct orthogonal designs.

Corollary 1 *Suppose there exists a weighing matrix $W(2n, 2n-k)$ constructed from two circulants, whose first rows have spread σ. Suppose there exist two directed sequences of length σ, with NPAF zero and type (σ, σ). Then there exist four directed sequences with PAF zero, that can be used in the Goethals-Seidel array to give an $OD(4n; 2\sigma, 2\sigma, 4n-2k)$.*

Proof

Construct the four sequences P, Q, R, S with PAF zero, that will be used in the Goethals-Seidel array, as in Theorem 3. We only need to show that these sequences are directed. Denote the two directed sequences of length σ, with NPAF zero and type (σ, σ) by $P' = [p_1, \cdots, p_\sigma]$ and $Q' = [q_1, \cdots, q_\sigma]$. We note that the term in the PAF of P which arises from the p_i of P' with the a_j of $[a_{\sigma+1}, \cdots, a_n]$ will appear as $p_i a_j$ or $a_j p_i$ and will be negated, by the term arising from the $-p_i$ of $-P'$ with the a_j of $[a_{\sigma+1}, \cdots, a_n]$ which appears as $-p_i a_j$ or $-a_j p_i$, respectively, in the PAF of Q. The construction ensures that these terms are canceled out without relying on commutativity of the involved variables. We have the same for Q' and the sequence $[b_{\sigma+1}, \cdots, b_n]$ in R and S.

Since the sequences P' and Q' are directed we have that the terms arising in the PAF of P from P' will be negated in the PAF of Q from $-P'$ without relying on commutativity. The same holds for Q' in the sequences R and S. Therefore, the sequences P, Q, R, S with PAF zero, are directed.

□

We illustrate the application of Corollary 1 with the following example:

Example 4 *Take $n = 11$, $k = 13$ and we begin with a spread-normalized weighing matrix $W(2\cdot 11, 2\cdot 11 - 13) = W(2\cdot 11, 9)$ constructed from two circulants with spread $\sigma = 2$*

```
A = [0, 0, 0, 0,  0, 1, 0, -1, 0, -1, 1]
B = [0, 0, 1, 0, -1, 0, 0,  1, 0,  1, 1]
```

Note that $\sigma = \min(s(A), s(B)) = \min(5,2) = 2$. *We multiply* A *and* B *by the commuting variable* α *to obtain:*

$$\begin{array}{rcl} A & = & [0,\ 0,\ 0,\ 0,\ 0,\ \alpha,\ 0,\ -\alpha,\ 0,\ -\alpha,\ \alpha] \\ B & = & [0,\ 0,\ \alpha,\ 0,\ -\alpha,\ 0,\ 0,\ \alpha,\ 0,\ \alpha,\ \alpha] \end{array}$$

We can take the directed sequences of length $\sigma = 2$, *with NPAF zero and type* $(u,v) = (2,2)$ *to be:*

$$[p_1, p_2] = [x, y] \text{ and } [q_1, q_2] = [x, -y].$$

Then the directed sequences P, Q, R, S *that will have PAF zero are:*

$$\begin{array}{ll} P & = [\ x,\ y,\ 0,\ 0,\ 0,\ \alpha,\ 0,\ -\alpha,\ 0,\ -\alpha,\ \alpha] \\ Q & = [\ -x,\ -y,\ 0,\ 0,\ 0,\ \alpha,\ 0,\ -\alpha,\ 0,\ -\alpha,\ \alpha] \\ R & = [\ x,\ -y,\ \alpha,\ 0,\ -\alpha,\ 0,\ 0,\ \alpha,\ 0,\ \alpha,\ \alpha] \\ S & = [\ -x,\ y,\ \alpha,\ 0,\ -\alpha,\ 0,\ 0,\ \alpha,\ 0,\ \alpha,\ \alpha] \end{array}$$

and they can be used in the Goethals-Seidel array to obtain an $OD(4n; 2u, 2v, 4n-2k)$, *i.e. an* $OD(44; 4, 4, 18)$.

We illustrate the application of Theorem 3 with the following example:

Example 5 *Take* $n = 11$, $k = 13$ *and we begin with a spread-normalized weighing matrix* $W(2\cdot 11, 2\cdot 11 - 13) = W(2\cdot 11, 9)$ *constructed from two circulants with spread* $\sigma = 3$

```
A = [0, 0, 0, 1, 0, -1, -1, -1, 0,  0, -1]
B = [0, 0, 0, 0, 1, -1,  0,  0, 0, -1,  1]
```

Note that $\sigma = \min(s(A), s(B)) = \min(3,4) = 3$. *We multiply* A *and* B *by the commuting variable* α *to obtain:*

$$\begin{array}{l} A = [0,\ 0,\ 0,\ \alpha,\ 0,\ -\alpha,\ -\alpha,\ -\alpha,\ 0,\ 0,\ -\alpha] \\ B = [0,\ 0,\ 0,\ 0,\ \alpha,\ -\alpha,\ 0,\ 0,\ 0,\ -\alpha,\ \alpha] \end{array}$$

We can take the sequences of length $\sigma = 3$, *with NPAF zero and type* $(u,v) = (1,4)$ *to be:*

$$[p_1, p_2, p_3] = [x, y, -x] \text{ and } [q_1, q_2, q_3] = [x, 0, x].$$

Then the sequences P, Q, R, S *that will have PAF zero are:*

$$\begin{array}{ll} P & = [\ x,\ y,\ -x,\ \alpha,\ 0,\ -\alpha,\ -\alpha,\ -\alpha,\ 0,\ 0,\ -\alpha] \\ Q & = [\ -x,\ -y,\ x,\ \alpha,\ 0,\ -\alpha,\ -\alpha,\ -\alpha,\ 0,\ 0,\ -\alpha] \\ R & = [\ x,\ 0,\ x,\ 0,\ \alpha,\ -\alpha,\ 0,\ 0,\ 0,\ -\alpha,\ \alpha] \\ S & = [\ -x,\ 0,\ -x,\ 0,\ \alpha,\ -\alpha,\ 0,\ 0,\ 0,\ -\alpha,\ \alpha] \end{array}$$

and they can be used in the Goethals-Seidel array to obtain an $OD(4n; 2u, 2v, 4n-2k)$, *i.e. an* $OD(44; 2, 8, 18)$.

Remark 1 *In example 5 the sequences P, Q, R, S with PAF zero, are not directed ($P_P(1)+P_Q(1)+P_R(1)+P_S(1) = 2(xy-yx) = 0$ if and only if x and y are commuting variables). This occurs, since the sequences $P' = [x, y, -x]$ and $Q' = [x, 0, x]$ with NPAF zero, are not directed ($N_{P'}(1) + N_{Q'}(1) = xy - yx = 0$ if and only if x and y are commuting variables).*

Theorem 3 has also a corollary pertaining to Golay sequences.

Corollary 2 *Suppose there exists a weighing matrix $W(2n, 2n-k)$ constructed from two circulants, whose first rows have spread σ, such that $\sigma \geq 2g$, where g is the length of a Golay sequence. Then there exist four directed sequences with PAF zero, that can be used in the Goethals-Seidel array to give an $OD(4n; 4g, 4g, 4n-2k)$.*

Proof

Mimicking the beginning of the proof of Theorem 3 we obtain:

$$\begin{aligned} A = & \ [\underbrace{0, \cdots, 0,}_{\sigma \text{ zeros}} \ a_{\sigma+1}, \ \cdots, \ a_n] \\ B = & \ [\underbrace{0, \cdots, 0,}_{\sigma \text{ zeros}} \ b_{\sigma+1}, \ \cdots, \ b_n] \end{aligned} \tag{3}$$

where $a_k, b_k \in \{0, \pm\alpha\}, k = \sigma+1, \ldots, n$ and either $a_{\sigma+1} \neq 0$ or $b_{\sigma+1} \neq 0$.
Now suppose the two Golay sequences of length g are G and H. Denote by $G^\star$ and $H^\star$ the reverse sequences of G and H respectively. Denote by 0_m a sequence of m consecutive zeros. The symbol $|$ denotes concatenation of sequences. Let x and y be commuting variables. Then the $OD(4n; 4g, 4g, 4n-2k)$ can be constructed by forming the four circulant matrices with the given first rows P, Q, R, S below, which are then used in the Goethals-Seidel array.

$$\begin{aligned} P = [& \quad Gx \mid & Hy \mid & \ 0_{\sigma-2g}, & a_{\sigma+1}, & \ \cdots, & a_n] \\ Q = [& \ -Gx \mid & -Hy \mid & \ 0_{\sigma-2g}, & a_{\sigma+1}, & \ \cdots, & a_n] \\ R = [& \quad H^\star x \mid & -G^\star y \mid & \ 0_{\sigma-2g}, & b_{\sigma+1}, & \ \cdots, & b_n] \\ S = [& \ -H^\star x \mid & G^\star y \mid & \ 0_{\sigma-2g}, & b_{\sigma+1}, & \ \cdots, & b_n] \end{aligned} \tag{4}$$

Note that $0_{\sigma-2g}$ is a (possibly empty) sequence of $\sigma - 2g$ zeros. The sequences P, Q, R, S have PAF zero. Indeed, denote $P' = [Gx \mid Hy]$ and $Q' = [H^\star x \mid -G^\star y]$, then P' and Q' have zero NPAF and the construction ensures these sequences are directed. Any products that arise in the PAF of P from elements of P' with the elements of the sequence $[a_{\sigma+1}, \cdots, a_n]$ are negated in the PAF of Q by the product of the elements of $-P'$ with the elements of the sequence $[a_{\sigma+1}, \cdots, a_n]$, without relying on the commutativity of the involved variables. We have the same for Q' and the sequence $[b_{\sigma+1}, \cdots, b_n]$ in R and S. The sum of the products from elements of the sequence $[a_{\sigma+1}, \cdots, a_n]$ with the products from elements of the sequence $[b_{\sigma+1}, \cdots, b_n]$ is equal to zero in the PAF of P, Q, R, S, since the sequences A and B have PAF zero. We have the same for the sum of the products from elements of P' with the products from elements of Q' in the PAF of P, Q, R, S, since these sequences have NPAF zero.

Since the sequences P' and Q' are directed we have that the terms arising in the PAF of P from P' will be negated in the PAF of Q from $-P'$ without relying on commutativity. The same holds for Q' in the sequences R and S. This gives the directed sequences P, Q, R, S with PAF zero required for an $OD(4n; 2g, 2g, 4n-2k)$.
□

We illustrate the application of Corollary 2 with the following example:

Example 6 *Take $n = 15$, $k = 21$ and we begin with a spread-normalized weighing matrix $W(2\cdot 15, 2\cdot 15 - 21) = W(2\cdot 15, 9)$ constructed from two circulants with spread $\sigma = 5$*

```
A = [0, 0, 0, 0, 0, 0, -1, 1, 0,  0, 0, 1, 0, 1,  1]
B = [0, 0, 0, 0, 0, 1,  0, 0, 0, -1, 0, 0, 1, 0, -1]
```

Note that $\sigma = \min(s(A), s(B)) = \min(6, 5) = 5$. We multiply A and B by the commuting variable α to obtain:

$$\begin{array}{l} A = [0,\ 0,\ 0,\ 0,\ 0,\ 0,\ -\alpha,\ \alpha,\ 0,\ 0,\ 0,\ \alpha,\ 0,\ \alpha,\ \alpha] \\ B = [0,\ 0,\ 0,\ 0,\ 0,\ \alpha,\ 0,\ 0,\ 0,\ -\alpha,\ 0,\ 0,\ \alpha,\ 0,\ -\alpha] \end{array}$$

We can take the Golay sequences of length $g = 2$, (where $2g = 4 < 5 = \sigma$) with NPAF zero to be:

$$G = [1, 1] \text{ and } H = [1, -1].$$

Then the reverse sequences $G^\star$ and $H^\star$ of G and H are:

$$G^\star = [1, 1] \text{ and } H^\star = [-1, 1].$$

The directed sequences $P' = [Gx \mid Hy]$ and $Q' = [H^\star x \mid -G^\star y]$ which have NPAF zero are:

$$P' = [x, x, y, -y] \text{ and } Q' = [-x, x, -y, -y].$$

Then the desired directed sequences P, Q, R, S that will have PAF zero are:

$$\begin{array}{lcl} P & = & [\ x,\ x,\ y,\ -y,\ 0,\ 0,\ -\alpha,\ \alpha,\ 0,\ 0,\ 0,\ \alpha,\ 0,\ \alpha,\ \alpha] \\ Q & = & [\ -x,\ -x,\ -y,\ y,\ 0,\ 0,\ -\alpha,\ \alpha,\ 0,\ 0,\ 0,\ \alpha,\ 0,\ \alpha,\ \alpha] \\ R & = & [\ -x,\ x,\ -y,\ -y,\ 0,\ \alpha,\ 0,\ 0,\ 0,\ -\alpha,\ 0,\ 0,\ \alpha,\ 0,\ -\alpha] \\ S & = & [\ x,\ -x,\ y,\ y,\ 0,\ \alpha,\ 0,\ 0,\ 0,\ -\alpha,\ 0,\ 0,\ \alpha,\ 0,\ -\alpha] \end{array}$$

and they can be used in the Goethals-Seidel array to obtain an $OD(4n; 4g, 4g, 4n-2k)$, i.e. an $OD(60; 8, 8, 18)$.

4 Orthogonal designs using directed sequences

The main advantage of directed sequences, their multiplication property, is that their variables can be replaced by sequences with NPAF zero to obtain longer sequences of different type, with NPAF zero, suitable for the construction of large orthogonal designs.

Unfortunately, we cannot replace all the variables of the four directed sequences produced in Corollaries 1 and 2 by other sequences with PAF zero since we generate three-variable orthogonal designs. However, we are able to replace the variables of the intermediate sequences used in the aforementioned corollaries. This is illustrated in the following example.

Example 7 *Take $n = 17$, $k = 25$ and we begin with a spread-normalized weighing matrix $W(2\cdot 17, 2\cdot 17-25) = W(2\cdot 17, 9)$ constructed from two circulants with spread $\sigma = 6$*

```
A = [0, 0, 0, 0, 0, 0, 1, 0,  1, 0, 0, 0,  0, 0, 1, 1, -1]
B = [0, 0, 0, 0, 0, 0, 0, 0, -1, 0, 0, 0, -1, 0, 1, 0,  1]
```

Note that $\sigma = \min(s(A), s(B)) = \min(6, 8) = 6$. We multiply A and B by the commuting variable α to obtain:

$$\begin{array}{rcl} A &=& [0,\ 0,\ 0,\ 0,\ 0,\ 0,\ \alpha,\ 0,\ \alpha,\ 0,\ 0,\ 0,\ 0,\ 0,\ \alpha,\ \alpha,\ -\alpha] \\ B &=& [0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ -\alpha,\ 0,\ 0,\ 0,\ -\alpha,\ 0,\ \alpha,\ 0,\ \alpha] \end{array}$$

We consider the following two directed sequences of length $m = 2$, with NPAF zero and type $(m, m) = (2, 2)$:

$$E = [c, d] \ \ and \ \ F = [c, -d].$$

Therefore we can replace the variables of the sequences E and F by the following sequences E' and F' of length $s = 3$ with NPAF zero and type $(u, v) = (1, 4)$ to obtain longer sequences with NPAF zero.

$$E' = [x, y, -x] \ \ and \ \ F' = [x, 0, x].$$

In particular by replacing the variables c, d in the sequences E, F with the sequences E', F' respectively we obtain the following set of sequences of length $2\cdot 3 = 6 = \sigma$ and type $(2\cdot 1, 2\cdot 4) = (2, 8)$ with NPAF zero:

$$[p_1, p_2, p_3, p_4, p_5, p_6] = [x, y, -x, x, 0, x], \ \ [q_1, q_2, q_3, q_4, q_5, q_6] = [x, y, -x, -x, 0, -x].$$

which can be used in the construction of Theorem 3, to give the sequences P, Q, R, S that will have PAF zero:

$$\begin{array}{l} P = [\ \ x,\ \ y,\ -x,\ \ x,\ 0,\ \ x,\ \alpha,\ 0,\ \ \alpha,\ 0,\ 0,\ 0,\ \ 0,\ 0,\ \alpha,\ \alpha,\ -\alpha] \\ Q = [\ -x,\ -y,\ \ x,\ -x,\ 0,\ -x,\ \alpha,\ 0,\ \ \alpha,\ 0,\ 0,\ 0,\ \ 0,\ 0,\ \alpha,\ \alpha,\ -\alpha] \\ R = [\ \ x,\ \ y,\ -x,\ -x,\ 0,\ -x,\ 0,\ 0,\ -\alpha,\ 0,\ 0,\ 0,\ -\alpha,\ 0,\ \alpha,\ 0,\ \alpha] \\ S = [\ -x,\ -y,\ \ x,\ \ x,\ 0,\ \ x,\ 0,\ 0,\ -\alpha,\ 0,\ 0,\ 0,\ -\alpha,\ 0,\ \alpha,\ 0,\ \alpha] \end{array}$$

and they can be used in the Goethals-Seidel array to obtain an $OD(4n; 2mu, 2mv, 4n-2k)$, i.e. an $OD(68; 4, 16, 18)$.

Corollary 3 *Suppose there exists a weighing matrix $W(2n, 2n-k)$ constructed from two circulants, whose first rows have spread σ. Suppose there exist two directed sequences of length m with NPAF zero and type (m, m). Furthermore, suppose there exist two sequences of length s, with NPAF zero and type (u, v). If $\sigma \geq ms$, then there exists an $OD(4n; 2mu, 2mv, 4n-2k)$.*

Proof
Mimicking the proof of Theorem 3 we only need to construct the sequences P' and Q' with NPAF zero of length σ. We distinguish between two cases. Firstly, if $\sigma = ms$ we replace the variables of the directed sequences with the sequences of length s with NPAF zero and type (u, v). The derived sequences are of length $ms = \sigma$ and type (mu, mv) with NPAF zero and are the desired sequences P' and Q'. Otherwise, if $\sigma > ms$ we construct the sequences of NPAF zero as previously, and in addition we pad them in the end with the sequence $0_{\sigma-ms}$ of $(\sigma - ms)$ consecutive zeros in order to achieve the required length σ for the sequences P' and Q'. Again, these sequences are the required sequences for the construction of Theorem 3 in order to give an $OD(4n; 2mu, 2mv, 4n-2k)$.

□

5 Corollaries for $W(2n, 2n-13)$ and $W(2n, 2n-17)$

We use the results of [4] for the next two corollaries.

Corollary 4 *Suppose there exists a $W(2n, 2n-13)$ constructed using two circulants with first rows of spread 6. Then there exist*

1. $OD(4n; 2, 2, 4n-26)$,
2. $OD(4n; 4, 4, 4n-26)$,
3. $OD(4n; 2, 8, 4n-26)$,
4. $OD(4n; 8, 8, 4n-26)$,
5. $OD(4n; 10, 10, 4n-26)$,

constructed using four circulant matrices in the Goethals-Seidel array.

Proof
For the cases 1) to 5) use the values given in the following table or other suitable values, to construct an $OD(4n; e, f, 4n-26)$:

e, f	p_1	p_2	p_3	p_4	p_5	p_6	q_1	q_2	q_3	q_4	q_5	q_6
2, 2	x	0	0	0	0	0	y	0	0	0	0	0
4, 4	x	y	0	0	0	0	x	$-y$	0	0	0	0
2, 8	x	y	$-x$	0	0	0	x	0	x	0	0	0
8, 8	x	x	y	$-y$	0	0	x	$-x$	y	y	0	0
10, 10	x	x	$-x$	y	0	y	$-x$	0	$-x$	$-y$	y	y

□

Corollary 5 *Suppose there exists a $W(2n, 2n-17)$ constructed using two circulants with first rows of spread 8. Then there exist*

1. $OD(4n; 2, 2, 4n-34)$,
2. $OD(4n; 4, 4, 4n-34)$,
3. $OD(4n; 2, 8, 4n-34)$,
4. $OD(4n; 8, 8, 4n-34)$,
5. $OD(4n; 10, 10, 4n-34)$,
6. $OD(4n; 16, 16, 4n-34)$,

constructed using four circulant matrices in the Goethals-Seidel array.

Proof
For the cases 1) to 6) use the values given in the following table or other suitable values, to construct an $OD(4n; e, f, 4n-34)$:

e, f	p_1	p_2	p_3	p_4	p_5	p_6	p_7	p_8	q_1	q_2	q_3	q_4	q_5	q_6	q_7	q_8
$2, 2$	x	0	0	0	0	0	0	0	y	0	0	0	0	0	0	0
$4, 4$	x	y	0	0	0	0	0	0	x	$-y$	0	0	0	0	0	0
$2, 8$	x	y	$-x$	0	0	0	0	0	x	0	x	0	0	0	0	0
$8, 8$	x	x	y	$-y$	0	0	0	0	x	$-x$	y	y	0	0	0	0
$10, 10$	x	x	$-x$	y	0	y	0	0	$-x$	0	$-x$	$-y$	y	y	0	0
$16, 16$	x	x	x	$-x$	y	y	$-y$	y	x	$-x$	x	x	y	$-y$	$-y$	$-y$

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