Novel Constructions of Improved Square Complex Orthogonal Designs for Eight Transmit Antennas

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***Abstract*—Constructions of square, maximum rate Complex Orthogonal Space-Time Block Codes (CO STBCs) are well known, however codes constructed via the known methods include numerous zeros, which impede their practical implementation. By modifying the Williamson and Wallis-Whiteman arrays to apply to complex matrices, we propose two methods of construction of *square*, order-$4n$ CO STBCs from *square*, order-n codes which satisfy certain properties. Applying the proposed methods, we construct *square*, *maximum rate*, order-8 CO STBCs with no zeros, such that the transmitted symbols are equally dispersed through transmit antennas. Those codes, referred to as the *improved square CO STBCs*, have the advantages that the power is equally transmitted via each transmit antenna during every symbol time slot and that a lower peak-to-average power ratio (PAPR) is required to achieve the same bit error rates as the conventional CO STBCs with zeros.**

***Index Terms*—AOD, CO STBC, MIMO, orthogonal design, PAPR, STBC.**

I. Introduction

Complex Orthogonal Space-Time Block Codes (CO STBCs) have been intensively examined, as they provide large transmit diversity and increase the capacity of wireless channels, while requiring a very simple maximum likelihood (ML) decoding method [3], [4], [5] [6] [7]. A $p \times n$ CO STBC over k variables is corresponding to n transmit (Tx) antennas, decoding delay (or memory length) of p, rate $R = k/p$ and is denoted as $[p, n, k]$ CO STBC. Given n and R, the goal is to minimize the decoding delay p. Hence, *square* CO STBCs are particularly interesting because they require the minimum processing delay (minimum memory length as well) for the same rate and the same number of Tx antennas. Another consideration for practical implementation is the number of zeros in a code. Compared to a code with fewer zeros, a code with more zeros results in a higher peak-to-average power ratio (PAPR), leading to the necessity of the use of circuits with the linear characteristic within a large dynamic range. Otherwise, the received signals may suffer from serious distortion. Having many zeros can also impede practical implementation, especially in high data rate wireless communication systems, since some Tx antennas must be turned off during transmission. Furthermore, it would be more practical if the power of signals can be equally transmitted via each Tx antenna during every symbol time slot (STS). Given the above considerations for CO STBCs, this paper focuses on constructing *square* CO STBCs with maximum rate, minimum decoding delay, no zero entries, and equal power transmission per Tx antenna during each STS.

The simplest square CO STBCs is the Alamouti code [3], which achieves a rate one for two Tx antennas. In contrast, square CO STBCs for more than two Tx antennas cannot achieve rate one [4], [8], but they can still achieve full diversity for a given number of Tx antennas. Constructions of *square* CO STBCs for a higher number of Tx antennas, e.g. 4 and 8, have been well examined in literature, such as [4] and [7]. The code $\mathbf{Z}_1$ in Eq. (1) [7] is one of the examples of the conventional square CO STBCs for 8 Tx antennas. The conventional structures yield square CO STBCs of *maximum rate*, which is, for instance, 1/2 for 8 Tx antennas. However, these maximum rate codes have many zero entries, which are undesirable.

It is important to clarify that, according to Liang's paper [4], the maximum achievable rate for CO STBCs of orders $n = 2m - 1$ or $n = 2m$ is (see Eq. (130) in [4]):

$$R_{max} = (m + 1)/2m. \tag{2}$$

However, note that this maximum rate is only achievable for *non-square* constructions, except for the special case when $m = 1$, i.e. when $n = 1$ or $n = 2$. For *square* constructions of orders $n = 2^a(2b + 1)$, where a and b are integers, the maximum achievable rate is:

$$R_{max} = (a + 1)/2^a(2b + 1). \tag{3}$$

When $m = 1$, (2) and (3) provide the same results. Readers should refer to Corollary 2 and Section II D in [4], or Section IV in [7] for more details.

Particularly, for $n = 8$, i.e., $m = 4$, $a = 3$ and $b = 0$, the maximum achievable rate of *non-square* CO STBCs following (2) is 5/8, while the maximum achievable rate of *square* CO STBCs according to (3) is 1/2 only. In Liang's paper, the authors made an *unclear* statement in the abstract that the achievable maximum rate for $n = 2m - 1$ and $n = 2m$ is $(m + 1)/2m$, but did not state if this maximum rate is achievable by *non-square* or *square* constructions. This easily makes readers confused, except when readers go deeply into the Liang's paper.

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$$\mathbf{Z}_1 = \begin{bmatrix} s_1 & s_2 & s_3 & 0 & s_4 & 0 & 0 & 0 \\ -s_2^* & s_1^* & 0 & -s_3 & 0 & -s_4 & 0 & 0 \\ -s_3^* & 0 & s_1^* & s_2 & 0 & 0 & -s_4 & 0 \\ 0 & s_3^* & -s_2^* & s_1 & 0 & 0 & 0 & s_4 \\ -s_4^* & 0 & 0 & 0 & s_1^* & s_2 & s_3 & 0 \\ 0 & s_4^* & 0 & 0 & -s_2^* & s_1 & 0 & -s_3 \\ 0 & 0 & s_4^* & 0 & -s_3^* & 0 & s_1 & s_2 \\ 0 & 0 & 0 & -s_4^* & 0 & s_3^* & -s_2^* & s_1^* \end{bmatrix} \tag{1}$$

Square CO STBCs have a great advantage over *non-square* CO STBCs that they require a much smaller length of the codes, i.e., much smaller processing delay, with the consequence of the slightly smaller maximum code rate compared to the achievable maximum code rate of *non-square* CO STBCs. To demonstrate this, let us consider CO STBCs for $n = 8$ Tx antennas. An example for this case is the *non-square* [112,8,70] CO STBC given in Appendix E in Liang's paper [4], that achieves the maximum rate 5/8 and requires the length of 112 STSs. It was proved later in [9] that the minimal length of complex orthogonal designs for 8 Tx antennas with the maximal rate 5/8 is 56, rather than 112. This observation has been confirmed by Liang in [10] where the *non-square* [56,8,35] CO STBC with the maximal rate 5/8 and minimal length 56 has been derived.

As opposed to *non-square* CO STBCs, *square* CO STBCs only require the length of 8 STSs to achieve the maximum rate 1/2, which is slightly smaller than the maximum rate of *non-square* CO STBCs. Clearly, *square* CO STBCs require a much shorter length, especially for a large number of Tx antennas, with the consequence of a slightly lower maximum code rate. For this reason, in this paper, we only consider *square* CO STBCs.

Square CO STBCs with no zero entries have been proposed in the literature, such as [3] and [6], for orders 2, 4. In [11], from *Amicable Orthogonal Designs* (AODs), we constructed two square, order 8 CO STBCs $\mathbf{Z}_2$ and $\mathbf{Z}_3$ (see Equations (4) and (5)) with *fewer zeros* than the conventional codes [4], [7]. The background knowledge on AODs can be found in [12]. Later, in [1] and [13], we constructed a square, order 8 CO STBC $\mathbf{Z_4}$ *without any zero*, which is given in (6) where $j = \sqrt{-1}$.

As pointed out in [1], [13], the entries z_{lk} $(l = 5, \ldots, 8, k = 1, \ldots, 8)$ of $\mathbf{Z_4}$ are composed of the real part of one indeterminate and the imaginary part of another indeterminate, e.g., $z_{51} = -s_4^R + js_3^I$. This observation means that if the indeterminates $s_1, \ldots, s_4$ are chosen from the complex signal constellations where s_i^R or s_i^I $(i = 1, \ldots, 4)$ can be equal to zero, e.g., the QPSK constellation $(1, -1, j, -j)$ then, some of the entries of the matrix $\mathbf{Z_4}$ can be equal to zero depending on the transmitted data. Therefore, such constellations should be avoided. An example of the constellation where the power is evenly spread among the Tx antennas independently of the transmitted data is the QPSK constellation $(1+j, 1-j, -1+j, -1-j)$.

The square CO STBC in (6) has the following advantages:

1) It is not required to turn off any Tx antenna during transmission, unlike in the conventional CO STBC [4], [7].
2) When the indeterminates are chosen from a suitable constellation, $\mathbf{Z_4}$ has no zero entries, hence, it requires a smaller peak power per Tx antenna to achieve the same BER as in the conventional square CO STBCs with zeros [4], [7]. Equivalently, it provides a better BER compared to the conventional square CO STBCs with the same peak power at Tx antennas.

Independently, also based on AODs, C. Yuen et al. [14] constructed a *solitary*, square, order-8 CO STBC with no zeros, which is referred to as $\mathbf{G}_8$ and is given in Eq. (7). This square CO STBC has an advantage over our code $\mathbf{Z_4}$ in that it does not require the restriction on signal constellations. However, it is always difficult to construct square CO STBCs based on AODs, especially for those codes of high orders, since various weighting matrices must be incorporated. For instance, to construct a square, *maximum rate* CO STBC of order 8, eight matrices of size 8 × 8 (4 weighting matrices for the real parts of variables and 4 other weighting matrices for the imaginary parts), which simultaneously satisfy several strong conditions of AODs [14], [12], [15], must be found.

In this paper, by modifying the Williamson and Wallis-Whiteman arrays to apply to complex matrices, we propose two *novel* methods of construction of *square*, order-$4n$ CO STBCs from *square*, order-n codes which satisfy certain properties. Applying the proposed methods, we construct *square*, *maximum rate*, order-8 CO STBCs with no zeros, such that the transmitted symbols equally disperse through Tx antennas. Besides having the maximum rate, the minimal decoding delay, and no zero entries, the resultant codes, referred to as the *improved square CO STBCs*, have the following practical advantages: a) They do not require any restriction on allowable signal constellations; b) It is possible to transmit symbols with equal power for any STS at any Tx antenna; and c) A lower peak power per Tx antenna is required to achieve the same bit error rates as for the conventional CO STBCs with zeros.

As mentioned in more details later in this paper, in order to construct, for instance, 8×8 CO STBCs, the *main* task in our methods is to find two sub-matrices of size 2×2 which satisfy certain properties, rather than finding 8 weighting matrices of size 8×8 simultaneously as in the AOD approaches, such as in [14]. More importantly, our methods give a transition from square, order-n CO STBCs satisfying certain properties to square, order-$4n$ CO STBCs. A good reference highly related to the topic of this paper is [16] where their constructions might, in some cases, result in a structure similar to one of

$$
\mathbf{Z}_2 = \begin{bmatrix}
s_1 & s_2 & \frac{s_3}{\sqrt{2}} & \frac{s_3}{\sqrt{2}} & 0 & 0 & \frac{s_4}{\sqrt{2}} & \frac{s_4}{\sqrt{2}} \\
-s_2^* & s_1^* & \frac{s_3}{\sqrt{2}} & -\frac{s_3}{\sqrt{2}} & 0 & 0 & \frac{s_4}{\sqrt{2}} & -\frac{s_4}{\sqrt{2}} \\
\frac{s_3^*}{\sqrt{2}} & \frac{s_3^*}{\sqrt{2}} & -s_1^R + js_2^I & -s_2^R + js_1^I & \frac{s_4}{\sqrt{2}} & \frac{s_4}{\sqrt{2}} & 0 & 0 \\
\frac{s_3^*}{\sqrt{2}} & -\frac{s_3^*}{\sqrt{2}} & s_2^R + js_1^I & -s_1^R - js_2^I & \frac{s_4}{\sqrt{2}} & -\frac{s_4}{\sqrt{2}} & 0 & 0 \\
0 & 0 & \frac{s_4^*}{\sqrt{2}} & \frac{s_4^*}{\sqrt{2}} & s_1 & s_2 & -\frac{s_3^*}{\sqrt{2}} & -\frac{s_3^*}{\sqrt{2}} \\
0 & 0 & \frac{s_4^*}{\sqrt{2}} & -\frac{s_4^*}{\sqrt{2}} & -s_2^* & s_1^* & -\frac{s_3^*}{\sqrt{2}} & \frac{s_3^*}{\sqrt{2}} \\
\frac{s_4^*}{\sqrt{2}} & \frac{s_4^*}{\sqrt{2}} & 0 & 0 & -\frac{s_3}{\sqrt{2}} & -\frac{s_3}{\sqrt{2}} & -s_1^R + js_2^I & -s_2^R + js_1^I \\
\frac{s_4^*}{\sqrt{2}} & -\frac{s_4^*}{\sqrt{2}} & 0 & 0 & -\frac{s_3}{\sqrt{2}} & \frac{s_3}{\sqrt{2}} & s_2^R + js_1^I & -s_1^R - js_2^I
\end{bmatrix} \tag{4}
$$

$$
\mathbf{Z}_3 = \begin{bmatrix}
s_1 & 0 & s_3^R + js_2^I & s_2^R + js_3^I & \frac{s_4}{2} & \frac{s_4}{2} & \frac{s_4}{2} & \frac{s_4}{2} \\
0 & s_1 & -s_2^R + js_3^I & s_3^R - js_2^I & \frac{s_4}{2} & -\frac{s_4}{2} & \frac{s_4}{2} & -\frac{s_4}{2} \\
-s_3^R + js_2^I & s_2^R + js_3^I & s_1^* & 0 & \frac{s_4}{2} & \frac{s_4}{2} & -\frac{s_4}{2} & -\frac{s_4}{2} \\
-s_2^R + js_3^I & -s_3^R - js_2^I & 0 & s_1^* & \frac{s_4}{2} & -\frac{s_4}{2} & -\frac{s_4}{2} & \frac{s_4}{2} \\
-\frac{s_4^*}{2} & -\frac{s_4^*}{2} & -\frac{s_4^*}{2} & -\frac{s_4^*}{2} & s_1^R - js_3^I & s_2^* & s_3^R - js_1^I & 0 \\
-\frac{s_4^*}{2} & \frac{s_4^*}{2} & -\frac{s_4^*}{2} & \frac{s_4^*}{2} & -s_2 & s_1^R + js_3^I & 0 & s_3^R - js_1^I \\
-\frac{s_4^*}{2} & -\frac{s_4^*}{2} & \frac{s_4^*}{2} & \frac{s_4^*}{2} & -s_3^R - js_1^I & 0 & s_1^R + js_3^I & -s_2^* \\
-\frac{s_4^*}{2} & \frac{s_4^*}{2} & \frac{s_4^*}{2} & -\frac{s_4^*}{2} & 0 & -s_3^R - js_1^I & s_2 & s_1^R - js_3^I
\end{bmatrix} \tag{5}
$$

$$
\mathbf{Z_4} = \begin{bmatrix}
s_1 & s_1 & s_2 & s_2 & s_3 & s_4 & s_3 & s_4 \\
s_1 & -s_1 & s_2 & -s_2 & s_4^* & -s_3^* & s_4^* & -s_3^* \\
s_2^* & s_2^* & -s_1^* & -s_1^* & s_3 & s_4 & -s_3 & -s_4 \\
s_2^* & -s_2^* & -s_1^* & s_1^* & s_4^* & -s_3^* & -s_4^* & s_3^* \\
-s_4^R + js_3^I & -s_3^R + js_4^I & -s_4^R + js_3^I & -s_3^R + js_4^I & s_2^R - js_1^I & s_2^R - js_1^I & s_1^R - js_2^I & s_1^R - js_2^I \\
-s_3^R - js_4^I & s_4^R + js_3^I & -s_3^R - js_4^I & s_4^R + js_3^I & s_2^R - js_1^I & -s_2^R + js_1^I & s_1^R - js_2^I & -s_1^R + js_2^I \\
-s_4^R + js_3^I & -s_3^R + js_4^I & s_4^R - js_3^I & s_3^R - js_4^I & s_1^R + js_2^I & s_1^R + js_2^I & -s_2^R - js_1^I & -s_2^R - js_1^I \\
-s_3^R - js_4^I & s_4^R + js_3^I & s_3^R + js_4^I & -s_4^R - js_3^I & s_1^R + js_2^I & -s_1^R - js_2^I & -s_2^R - js_1^I & s_2^R + js_1^I
\end{bmatrix} \tag{6}
$$

the structures mentioned in this paper. However, the structures reported there were gained via AODs while they are constructed, in this paper, via an independent approach, namely the submatrices-based design approach.

The paper is organized as follows. In Section II, we provide definitions and notations used throughout the paper. In Section III, we propose two methods for constructing high-rate, square CO STBCs of order $N = 4n$ from sub-matrices of order n. In Section IV, we use the proposed methods to construct square, *maximum rate*, order 8 CO STBCs, which are superior in several aspects to other known codes to date. Some simulation results are given in Section V. The paper is concluded by Section VI.

II. Definitions and Notations

Our proposed constructions in this paper are based on the following matrices, which are the variations of the Williamson and Wallis-Whiteman arrays mentioned in [12] (pp. 121 and 99, respectively), modified to apply to complex matrices

$$
\mathcal{O}_1 = \begin{bmatrix}
\mathbf{A} & \mathbf{B} & \mathbf{C} & \mathbf{D} \\
-\mathbf{B} & \mathbf{A} & \mathbf{D}^* & -\mathbf{C}^* \\
-\mathbf{C} & -\mathbf{D}^* & \mathbf{A} & \mathbf{B}^* \\
-\mathbf{D} & \mathbf{C}^* & -\mathbf{B}^* & \mathbf{A}
\end{bmatrix}, \tag{8}
$$

$$
\mathcal{O}_2 = \begin{bmatrix}
\mathbf{A} & \mathbf{B} & \mathbf{C} & \mathbf{D} \\
-\mathbf{B}^* & \mathbf{A}^* & -\mathbf{D} & \mathbf{C} \\
-\mathbf{C} & \mathbf{D}^* & \mathbf{A} & -\mathbf{B}^* \\
-\mathbf{D}^* & -\mathbf{C} & \mathbf{B} & \mathbf{A}^*
\end{bmatrix}, \tag{9}
$$

where $(.)^*$ denotes the element-wise conjugation if the argument is a matrix or a vector, or simply the complex conjugation if the argument is a complex variable. This means that $\mathbf{X}^*$ of a matrix $\mathbf{X}$ can be expressed as $\mathbf{X}^* = (\mathbf{X}^H)^T$. We denote $(.)^H$ to be the Hermitian transposition, while $(.)^T$ denotes the transposition (but not conjugate). $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$ and $\mathbf{D}$ are $n \times n$,

$$\mathbf{G}_8 = \begin{bmatrix} s_1^* & s_1^* & s_2 & -s_2 & s_3 & -s_3 & s_4 & -s_4 \\ js_1 & -js_1 & js_2^* & js_2^* & js_3^* & js_3^* & js_4^* & js_4^* \\ -s_2 & s_2 & s_1^* & s_1^* & s_4^* & -s_4^* & -s_3^* & s_3^* \\ -js_2^* & -js_2^* & js_1 & -js_1 & js_4 & js_4 & -js_3 & -js_3 \\ -s_3 & s_3 & -s_4^* & s_4^* & s_1^* & s_1^* & s_2^* & -s_2^* \\ -js_3^* & -js_3^* & -js_4 & -js_4 & js_1 & -js_1 & js_2 & js_2 \\ -s_4 & s_4 & s_3^* & -s_3^* & -s_2^* & s_2^* & s_1^* & s_1^* \\ -js_4^* & -js_4^* & js_3 & js_3 & -js_2 & -js_2 & js_1 & -js_1 \end{bmatrix} \quad (7)$$

square, orthogonal matrices of complex variables. Hence, $\mathcal{O}_1$ and $\mathcal{O}_2$ are $4n \times 4n$ matrices of complex variables.

Let $\mathcal{O}$ be a general notation representing either $\mathcal{O}_1$ or $\mathcal{O}_2$. Define $N = 4n$ and present N as $N = 2^a(2b+1)$, where a and b are integers. Let $\mu(N)$ be the maximum number of variables in $\mathcal{O}$. It is well known that the maximum number of variables in the *square* CO STBC of order N is $\mu(N) = a+1$. Readers may refer to [7], [12], or Corollary 2 in [4] for more details. Let μ_A, μ_B, μ_C and μ_D be the number of variables in $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, and $\mathbf{D}$, respectively.

Let U and $\mathcal{I}_U$ be the set of all variables in $\mathcal{O}$ and the set of all indices of elements in U, respectively. Similarly, let

$$\begin{aligned} U_1 &= \{s_{A1}, s_{A2}, \ldots, s_{A\mu_A}\}; U_2 = \{s_{B1}, s_{B2}, \ldots, s_{B\mu_B}\}; \\ U_3 &= \{s_{C1}, s_{C2}, \ldots, s_{C\mu_C}\}; U_4 = \{s_{D1}, s_{D2}, \ldots, s_{D\mu_D}\}; \end{aligned} \quad (10)$$

be the sets of variables in $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, and $\mathbf{D}$, respectively, and let $\mathcal{I}_{U_i}$, for $i = 1, \ldots, 4$, be the sets of indices of variables in the sub-matrices $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, and $\mathbf{D}$, respectively.

We require that the sub-matrices $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, and $\mathbf{D}$ satisfy:

$$\begin{cases} \bigcup U_i = U & i = 1, \ldots, 4 \\ \bigcap U_i U_j = \emptyset & i \neq j \end{cases} \quad (11)$$

where $\emptyset$ is the empty set. With the condition (11), clearly, if $\mathcal{O}$ comprises the maximum number of variables, we have

$$\mu_A + \mu_B + \mu_C + \mu_D = \mu(N). \quad (12)$$

It is noted that there is no predefined condition on μ_A, μ_B, μ_C and μ_D in order to achieve the upperbound $\mu(N)$. Instead, it is really flexible to select the set of μ_A, μ_B, μ_C and μ_D in order to achieve the equality $\mu_A + \mu_B + \mu_C + \mu_D = \mu(N)$, and a good choice of the set of μ_A, μ_B, μ_C and μ_D will lead to the optimal code structure. Some of such choices will be mentioned later in the examples within this paper.

Since $\mathbf{A}$ is a matrix on variables $\{s_{A1}, s_{A2}, \ldots, s_{A\mu_A}\}$, we define the vector $\mathbf{s_A} = (s_{A1}, s_{A2}, \ldots, s_{A\mu_A})$, and write:

$$\mathbf{A} = \mathbf{A}(\mathbf{s_A}) = \mathbf{A}(s_{A1}, s_{A2}, \ldots, s_{A\mu_A}).$$

Similarly, we denote the matrices $\mathbf{B}$, $\mathbf{C}$ and $\mathbf{D}$ as

$$\begin{aligned} \mathbf{B} &= \mathbf{B}(\mathbf{s_B}) = \mathbf{B}(s_{B1}, s_{B2}, \ldots, s_{B\mu_B}), \\ \mathbf{C} &= \mathbf{C}(\mathbf{s_C}) = \mathbf{C}(s_{C1}, s_{C2}, \ldots, s_{C\mu_C}), \\ \mathbf{D} &= \mathbf{D}(\mathbf{s_D}) = \mathbf{D}(s_{D1}, s_{D2}, \ldots, s_{D\mu_D}). \end{aligned} \quad (13)$$

For simplicity of notation, we sometimes write, for example, $\mathbf{A}_{\mathbf{s_A}}$ to represent $\mathbf{A}(\mathbf{s_A})$. Recall that the matrix $\mathbf{A}^*$ is derived from $\mathbf{A}$ by replacing each variable s_{Ai}, for $1 \leq i \leq \mu_A$, by its conjugate, i.e.

$$\mathbf{A}^* = \mathbf{A}(\mathbf{s_A}^*) = \mathbf{A}(s_{A1}^*, s_{A2}^*, \ldots, s_{A\mu_A}^*).$$

We can represent $\mathbf{B}^*$, $\mathbf{C}^*$, and $\mathbf{D}^*$ in a similar manner.

We state that a matrix $\mathbf{X}(\mathbf{s_X})$ is *of similar form* to a matrix $\mathbf{Y}(\mathbf{s_Y})$ (or just $\mathbf{X}$ is *of similar form* to $\mathbf{Y}$, for short) if $\mathbf{X} = k_X \mathbf{Y}(\mathbf{s_X})$, where $\mathbf{s_X}$ is a vector containing distinct complex variables $s_{X1}, s_{X2}, \ldots, s_{X\mu_X}$, and similarly, $\mathbf{s_Y}$ is a vector containing distinct complex variables $s_{Y1}, s_{Y2}, \ldots, s_{Y\mu_Y}$, and k_X is an arbitrary, non-zero, real coefficient. In this notation, we stipulate that the number of variables μ_X in $\mathbf{X}$ is at most equal to the number of variables μ_Y in $\mathbf{Y}$. To illustrate an example with $\mu_X = \mu_Y = 2$, $\mathbf{X}(\mathbf{s_X})$=$\begin{bmatrix} s_{X1} & s_{X2} \\ -s_{X2}^* & s_{X1}^* \end{bmatrix}$ (which presents the Alamouti code with two variables) is of similar form to $\mathbf{Y}(\mathbf{s_Y})$=$\begin{bmatrix} s_{Y1} & s_{Y2} \\ -s_{Y2}^* & s_{Y1}^* \end{bmatrix}$, since $\mathbf{X} = \mathbf{Y}(\mathbf{s_X}) = \mathbf{Y}(s_{X1}, s_{X2})$. To illustrate the case where $\mu_X = 1$ and $\mu_Y = 2$, $\mathbf{X}(\mathbf{s_X})$=$\begin{bmatrix} s_{X1} & s_{X1} \\ -s_{X1}^* & s_{X1}^* \end{bmatrix}$ (which presents the Alamouti code with only one variable) is also of similar form to $\mathbf{Y}$ since $\mathbf{X} = \mathbf{Y}(\mathbf{s_X}) = \mathbf{Y}(s_{X1})$.

By this notation, when we state that the matrix $\mathbf{C}$ in (13) is of similar form to the matrix $\mathbf{B}$, for instance, we imply that $\mathbf{C}$ can be represented as $\mathbf{C} = k_C \mathbf{B}(\mathbf{s_C})$ where the number of complex variables μ_C in $\mathbf{C}$ is at most equal to the number of complex variables μ_B in $\mathbf{B}$, i.e., $\mu_C \leq \mu_B$.

Finally, we denote $\mathbf{I}_n$ to be an identity matrix of order n.

III. Design Methods

In this section, we provide two new methods to construct square CO STBCs. In each case, we use sub-matrices of order n to build CO STBCs of order $N = 4n$. Our methods generalize the Williamson and Wallis-Whiteman arrays, which were originally used to build real orthogonal designs [12] (pp. 121 and 99, respectively).

Theorem 1: If the sub-matrices $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$ and $\mathbf{D}$ of order n satisfy the following necessary conditions:

1) $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, and $\mathbf{D}$ are orthogonal themselves and:

$$\mathbf{A}^H\mathbf{A} + \mathbf{B}^H\mathbf{B} + \mathbf{C}^H\mathbf{C} + \mathbf{D}^H\mathbf{D} = \sum_{i \in \mathcal{I}_U} l_i |s_i|^2 \mathbf{I}_n, \quad (14)$$

where l_i are definitely positive, real coefficients, and the complex variables s_i may be in U_1, U_2, U_3 or U_4 which are defined in (10).

2) The matrices $\mathcal{O}' = \begin{bmatrix} \mathbf{A} & \mathbf{B} \\ -\mathbf{B} & \mathbf{A} \end{bmatrix}$ and $\mathcal{O}'' = \begin{bmatrix} \mathbf{A} & \mathbf{B}^* \\ -\mathbf{B}^* & \mathbf{A} \end{bmatrix}$ are square Complex Orthogonal Designs (COD) of order $2n$.
3) $\mathbf{B}_{\mathbf{s}}^H\mathbf{B}_{\mathbf{s}'}$ and $\mathbf{B}_{\mathbf{s}}^T\mathbf{B}_{\mathbf{s}'}$ are symmetric for any possible pair of vectors $\mathbf{s}$ and $\mathbf{s}'$ of complex variables, where $\mathbf{B}_{\mathbf{s}}$ and $\mathbf{B}_{\mathbf{s}'}$ are shorthand for $\mathbf{B}(\mathbf{s})$ and $\mathbf{B}(\mathbf{s}')$, respectively.
4) $\mathbf{C}$ and $\mathbf{D}$ are of similar form to $\mathbf{B}$ and $\mathbf{B}$, respectively, $\mathbf{B}^*$ and $\mathbf{B}$, respectively, $\mathbf{B}$ and $\mathbf{B}^*$ respectively, or $\mathbf{B}^*$ and $\mathbf{B}^*$, respectively, i.e., $\mathbf{C}$ and $\mathbf{D}$ can be presented as one of the following forms

$$\begin{cases} \mathbf{C} = k_C\mathbf{B}(\mathbf{s}_{\mathbf{C}}) \\ \mathbf{D} = k_D\mathbf{B}(\mathbf{s}_{\mathbf{D}}) \end{cases} \quad \begin{cases} \mathbf{C} = k_C\mathbf{B}(\mathbf{s}_{\mathbf{C}}^*) \\ \mathbf{D} = k_D\mathbf{B}(\mathbf{s}_{\mathbf{D}}) \end{cases}$$
$$\begin{cases} \mathbf{C} = k_C\mathbf{B}(\mathbf{s}_{\mathbf{C}}) \\ \mathbf{D} = k_D\mathbf{B}(\mathbf{s}_{\mathbf{D}}^*) \end{cases} \quad \begin{cases} \mathbf{C} = k_C\mathbf{B}(\mathbf{s}_{\mathbf{C}}^*) \\ \mathbf{D} = k_D\mathbf{B}(\mathbf{s}_{\mathbf{D}}^*) \end{cases} \tag{15}$$

where k_C and k_D are arbitrary (positive or negative), real coefficients, and $\mu_C \leq \mu_B$, $\mu_D \leq \mu_B$

then

$$\mathcal{O} = \begin{bmatrix} \mathbf{A} & \mathbf{B} & \mathbf{C} & \mathbf{D} \\ -\mathbf{B} & \mathbf{A} & \mathbf{D}^* & -\mathbf{C}^* \\ -\mathbf{C} & -\mathbf{D}^* & \mathbf{A} & \mathbf{B}^* \\ -\mathbf{D} & \mathbf{C}^* & -\mathbf{B}^* & \mathbf{A} \end{bmatrix} \tag{16}$$

is a CO STBC of order $N = 4n$. If all coefficients $l_i = 1$, for $i \in \mathcal{I}_U$, then $\mathcal{O}$ is called square CO STBC without Linear Processing (LP) (or just square CO STBC for short). Otherwise, $\mathcal{O}$ is considered as a square CO STBC with LP. If $(\mu_A + \mu_B + \mu_C + \mu_D) = \mu(N)$, then $\mathcal{O}$ is a square, maximum rate CO STBC of order $4n$.

Proof: We prove Theorem 1 for the case that $\mathbf{C}$ and $\mathbf{D}$ are of similar form to $\mathbf{B}$ and $\mathbf{B}$, respectively. Similar arguments can be applied to three other cases. From (16), we have Eq. (17), where $\mathcal{L}$ in the matrix $\mathbf{M}$ denotes the lower triangular part under the main diagonal whose elements are the Hermitian transposes of the corresponding elements in the upper triangular part. For instance, we have the element $\mathcal{L}(2,1) = \mathbf{B}^H\mathbf{A} - \mathbf{A}^H\mathbf{B} + \mathbf{D}^{*H}\mathbf{C} - \mathbf{C}^{*H}\mathbf{D}$.

First, we prove the following equalities

$$\mathbf{B}^{*H}\mathbf{B}^* = \mathbf{B}^H\mathbf{B}, \tag{18}$$
$$\mathbf{C}^{*H}\mathbf{C}^* = \mathbf{C}^H\mathbf{C}, \tag{19}$$
$$\mathbf{D}^{*H}\mathbf{D}^* = \mathbf{D}^H\mathbf{D}. \tag{20}$$

Since $\mathbf{B}$ is orthogonal, we have

$$\mathbf{B}^H\mathbf{B} = \mathbf{B}\mathbf{B}^H = \sum_{i\in\mathcal{I}_{U_2}} l_i|s_i|^2\mathbf{I}_n,$$

which implies that $\mathbf{B}^H\mathbf{B}$ is a real, diagonal matrix and therefore

$$\mathbf{B}^H\mathbf{B} = [(\mathbf{B}^H\mathbf{B})^T]^H. \tag{21}$$

Using Eq. (21), it follows that

$$\mathbf{B}^H\mathbf{B} = [\mathbf{B}^T\mathbf{B}^*]^H = \mathbf{B}^{*H}(\mathbf{B}^T)^H = \mathbf{B}^{*H}\mathbf{B}^*.$$

Therefore, (18) has been proved. The same arguments can be applied to prove (19) and (20). Hence, if $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$ and $\mathbf{D}$ are orthogonal themselves and satisfy (14), then all elements (i.e. sub-matrices) on the main diagonal of the matrix $\mathbf{M} = \mathcal{O}^H\mathcal{O}$ are equal to

$$\mathbf{A}^H\mathbf{A} + \mathbf{B}^H\mathbf{B} + \mathbf{C}^H\mathbf{C} + \mathbf{D}^H\mathbf{D} = \sum_{i\in\mathcal{I}_U} l_i|s_i|^2\mathbf{I}_n.$$

Second, we prove the following equalities

$$\mathbf{A}^H\mathbf{B} - \mathbf{B}^H\mathbf{A} = \mathbf{O}_n, \tag{22}$$
$$\mathbf{A}^H\mathbf{C} - \mathbf{C}^H\mathbf{A} = \mathbf{O}_n, \tag{23}$$
$$\mathbf{A}^H\mathbf{D} - \mathbf{D}^H\mathbf{A} = \mathbf{O}_n, \tag{24}$$

where $\mathbf{O}_n$ is a zero matrix of order n. Eq. (22) holds as $\mathcal{O}'$ is a COD. Additionally, because $\mathbf{C}$ and $\mathbf{D}$ are of similar form to $\mathbf{B}$ (see (15)), the equalities (23) and (24) are straightforwardly proved (multiplication with real coefficients k_C and k_D does not change the property (22)).

Third, we prove the following equalities

$$\mathbf{B}^H\mathbf{C}^* - \mathbf{C}^H\mathbf{B}^* = \mathbf{O}_n, \tag{25}$$
$$\mathbf{B}^H\mathbf{D}^* - \mathbf{D}^H\mathbf{B}^* = \mathbf{O}_n, \tag{26}$$
$$\mathbf{C}^H\mathbf{D}^* - \mathbf{D}^H\mathbf{C}^* = \mathbf{O}_n. \tag{27}$$

Since $\mathbf{B}_{\mathbf{s}}^T\mathbf{B}_{\mathbf{s}'}$ is symmetric for any pair of vectors $\mathbf{s}$ and $\mathbf{s}'$ of complex variables, it follows that $(\mathbf{B}_{\mathbf{s}}^T\mathbf{B}_{\mathbf{s}'})^H \equiv \mathbf{B}_{\mathbf{s}'}^H\mathbf{B}_{\mathbf{s}}^{H^T}$ is also symmetric. Using this symmetry, it follows that

$$\mathbf{B}_{\mathbf{s}'}^H\mathbf{B}_{\mathbf{s}}^{H^T} = [\mathbf{B}_{\mathbf{s}'}^H\mathbf{B}_{\mathbf{s}}^{H^T}]^T \quad \Leftrightarrow \quad \mathbf{B}_{\mathbf{s}'}^H\mathbf{B}_{\mathbf{s}}^* = \mathbf{B}_{\mathbf{s}}^H\mathbf{B}_{\mathbf{s}'}^{H^T},$$
$$\Leftrightarrow \quad \mathbf{B}_{\mathbf{s}'}^H\mathbf{B}_{\mathbf{s}}^* = \mathbf{B}_{\mathbf{s}}^H\mathbf{B}_{\mathbf{s}'}^*.$$

In other words, we have

$$\mathbf{B}_{\mathbf{s}'}^H\mathbf{B}_{\mathbf{s}}^* - \mathbf{B}_{\mathbf{s}}^H\mathbf{B}_{\mathbf{s}'}^* = \mathbf{O}_n, \tag{28}$$

for any pair of vectors $\mathbf{s}$ and $\mathbf{s}'$. Due to the fact that $\mathbf{C}$ and $\mathbf{D}$ are of similar form to $\mathbf{B}$, by replacing $\mathbf{B}_{\mathbf{s}}$ and $\mathbf{B}_{\mathbf{s}'}$ in (28) by $\mathbf{B}$, $\mathbf{C}$ or $\mathbf{D}$, the equalities (25), (26) and (27) are proved.

From (22)–(27), we see that the elements $\mathbf{M}(1,2)$, $\mathbf{M}(1,3)$ and $\mathbf{M}(1,4)$ of the matrix $\mathbf{M} = \mathcal{O}^H\mathcal{O}$ are zero matrices.

Fourth, we prove the following equalities

$$\mathbf{B}^H\mathbf{C} - \mathbf{C}^{*H}\mathbf{B}^* = \mathbf{O}_n, \tag{29}$$
$$\mathbf{B}^H\mathbf{D} - \mathbf{D}^{*H}\mathbf{B}^* = \mathbf{O}_n, \tag{30}$$
$$\mathbf{C}^H\mathbf{D} - \mathbf{D}^{*H}\mathbf{C}^* = \mathbf{O}_n. \tag{31}$$

Due to $\mathbf{B}_{\mathbf{s}}^H\mathbf{B}_{\mathbf{s}'}$ being symmetric, the following equalities hold

$$\mathbf{B}_{\mathbf{s}}^H\mathbf{B}_{\mathbf{s}'} = [\mathbf{B}_{\mathbf{s}}^H\mathbf{B}_{\mathbf{s}'}]^T \quad \Leftrightarrow \quad \mathbf{B}_{\mathbf{s}}^H\mathbf{B}_{\mathbf{s}'} = \mathbf{B}_{\mathbf{s}'}^T\mathbf{B}_{\mathbf{s}}^*,$$
$$\Leftrightarrow \quad \mathbf{B}_{\mathbf{s}}^H\mathbf{B}_{\mathbf{s}'} = \mathbf{B}_{\mathbf{s}'}^{*\,H}\mathbf{B}_{\mathbf{s}}^*,$$
$$\Leftrightarrow \quad \mathbf{B}_{\mathbf{s}}^H\mathbf{B}_{\mathbf{s}'} - \mathbf{B}_{\mathbf{s}'}^{*\,H}\mathbf{B}_{\mathbf{s}}^* = \mathbf{O}_n \tag{32}$$

for any pair of vectors $\mathbf{s}$ and $\mathbf{s}'$. Due to $\mathbf{C}$ and $\mathbf{D}$ being of similar form to $\mathbf{B}$, by replacing $\mathbf{B}_{\mathbf{s}}$ and $\mathbf{B}_{\mathbf{s}'}$ in (32) by $\mathbf{B}$, $\mathbf{C}$ or $\mathbf{D}$, the equalities (29)–(31) are proved.

Finally, we prove that

$$\mathbf{A}^H\mathbf{B}^* - \mathbf{B}^{*H}\mathbf{A} = \mathbf{O}_n, \tag{33}$$
$$\mathbf{A}^H\mathbf{C}^* - \mathbf{C}^{*H}\mathbf{A} = \mathbf{O}_n, \tag{34}$$
$$\mathbf{A}^H\mathbf{D}^* - \mathbf{D}^{*H}\mathbf{A} = \mathbf{O}_n. \tag{35}$$

$$\begin{aligned}\mathbb{M} &= \mathcal{O}^H\mathcal{O}\\ &= \begin{bmatrix} A^HA+B^HB+C^HC+D^HD & A^HB-B^HA+C^H\bar{D}-D^H\bar{C} & A^HC-C^HA-B^H\bar{D}+D^H\bar{B} & A^HD-D^HA+B^H\bar{C}-C^H\bar{B}\\ & A^HA+B^HB+\bar{C}^H\bar{C}+\bar{D}^H\bar{D} & B^HC-\bar{C}^H\bar{B}+A^H\bar{D}-\bar{D}^HA & B^HD-\bar{D}^H\bar{B}-A^H\bar{C}+\bar{C}^HA\\ & & A^HA+\bar{B}^H\bar{B}+C^HC+\bar{D}^H\bar{D} & C^HD-\bar{D}^H\bar{C}+A^H\bar{B}-\bar{B}^HA\\ \mathcal{L} & & & A^HA+\bar{B}^H\bar{B}+\bar{C}^H\bar{C}+D^HD\end{bmatrix}.\end{aligned} \tag{17}$$

Eq. (33) holds since $\mathcal{O}''$ is a COD. Because $\mathbf{C}$ and $\mathbf{D}$ are of similar form to $\mathbf{B}$, by replacing $\mathbf{B}$ in (33) by $\mathbf{C}$ or $\mathbf{D}$, the equalities (34) and (35) are proved.

From (29)–(31) and (33)–(35), it follows that the elements $\mathbf{M}(2,3) = \mathbf{M}(2,4) = \mathbf{M}(3,4) = \mathbf{O}_n$. Since the lower triangular part $\mathcal{L}$ is the Hermitian transpose of the upper part, all elements in $\mathcal{L}$ are also zero matrices. Hence, $\mathbf{M}$ can be presented as

$$\mathbf{M} = \sum_{i\in\mathcal{I}_U} l_i|s_i|^2 diag(\mathbf{I}_n,\mathbf{I}_n,\mathbf{I}_n,\mathbf{I}_n) = \sum_{i\in\mathcal{I}_U} l_i|s_i|^2\mathbf{I}_N,$$

where $diag$ denotes a diagonal matrix. In other words, the matrix $\mathcal{O}$ in (16) is a square COD (also CO STBC) of order N=$4n$ with $(\mu_A+\mu_B+\mu_C+\mu_D)$ variables. Note that, if $\mathcal{O}$ comprises the maximum number of variables, i.e., Eq. (12) is satisfied, then $\mathcal{O}$ is a square, maximum rate CO STBC of order $4n$. Theorem 1 has been proved. ■

Similarly, we derived the following theorem, which is a variation of the Wallis-Whiteman array [12] (pp. 99), modified to apply to complex matrices:

Theorem 2: If the sub-matrices $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$ and $\mathbf{D}$ of order n satisfy the following necessary conditions:

1) $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$ and $\mathbf{D}$ are orthogonal themselves and
$$\mathbf{A}^H\mathbf{A}+\mathbf{B}^H\mathbf{B}+\mathbf{C}^H\mathbf{C}+\mathbf{D}^H\mathbf{D} = \sum_{i\in\mathcal{I}_U} l_i|s_i|^2\mathbf{I}_n,$$
where l_i are definitely positive, real coefficients, and the complex variables s_i may be in U_1, U_2, U_3 or U_4, which are defined in (10).
2) The matrices $\mathcal{O}' = \begin{bmatrix}\mathbf{C} & \mathbf{A}\\ -\mathbf{A} & \mathbf{C}\end{bmatrix}$ and $\mathcal{O}'' = \begin{bmatrix}\mathbf{C} & \mathbf{A}^*\\ -\mathbf{A}^* & \mathbf{C}\end{bmatrix}$ are square Complex Orthogonal Designs (COD) of order $2n$.
3) $\mathbf{A}_{\mathbf{s}}^H\mathbf{A}_{\mathbf{s}'}$ and $\mathbf{A}_{\mathbf{s}}^T\mathbf{A}_{\mathbf{s}'}$ are symmetric for any possible pair of vectors $\mathbf{s}$ and $\mathbf{s}'$ of complex variables, where $\mathbf{A}_{\mathbf{s}}$ and $\mathbf{A}_{\mathbf{s}'}$ are shorthand for $\mathbf{A}(\mathbf{s})$ and $\mathbf{A}(\mathbf{s}')$, respectively.
4) $\mathbf{B}$ and $\mathbf{D}$ are of similar form to $\mathbf{A}$ and $\mathbf{A}$, respectively, $\mathbf{A}^*$ and $\mathbf{A}$, respectively, $\mathbf{A}$ and $\mathbf{A}^*$ respectively, or $\mathbf{A}^*$ and $\mathbf{A}^*$, respectively, i.e., $\mathbf{B}$ and $\mathbf{D}$ can be presented as one of the following forms:
$$\begin{cases}\mathbf{B} = k_B\mathbf{A}(\mathbf{s_B})\\ \mathbf{D} = k_D\mathbf{A}(\mathbf{s_D})\end{cases} \quad \begin{cases}\mathbf{B} = k_B\mathbf{A}(\mathbf{s}_\mathbf{B}^*)\\ \mathbf{D} = k_D\mathbf{A}(\mathbf{s_D})\end{cases}$$
$$\begin{cases}\mathbf{B} = k_B\mathbf{A}(\mathbf{s_B})\\ \mathbf{D} = k_D\mathbf{A}(\mathbf{s}_\mathbf{D}^*)\end{cases} \quad \begin{cases}\mathbf{B} = k_B\mathbf{A}(\mathbf{s}_\mathbf{B}^*)\\ \mathbf{D} = k_D\mathbf{A}(\mathbf{s}_\mathbf{D}^*)\end{cases}$$
where k_B and k_D are arbitrary (positive or negative), real coefficients, and $\mu_B \le \mu_A$, $\mu_D \le \mu_A$

then

$$\mathcal{O} = \begin{bmatrix}\mathbf{A} & \mathbf{B} & \mathbf{C} & \mathbf{D}\\ -\mathbf{B}^* & \mathbf{A}^* & -\mathbf{D} & \mathbf{C}\\ -\mathbf{C} & \mathbf{D}^* & \mathbf{A} & -\mathbf{B}^*\\ -\mathbf{D}^* & -\mathbf{C} & \mathbf{B} & \mathbf{A}^*\end{bmatrix} \tag{36}$$

is a CO STBC of order $N = 4n$. If all coefficients $l_i = 1$ for $i \in \mathcal{I}_U$, then $\mathcal{O}$ is called square CO STBC without Linear Processing (LP) (or just square CO STBC for short). Otherwise, $\mathcal{O}$ is considered as a square CO STBC with LP. If $(\mu_A+\mu_B+\mu_C+\mu_D) = \mu(N)$, then $\mathcal{O}$ is a square, maximum rate CO STBC of order $4n$.

Proof: The proof of Theorem 2 is similar to the proof of Theorem 1. ■

IV. Examples of Maximum Rate, Square, Order-8 CO STBCs with No Zero Entries

In order to construct 8×8 CO STBCs of maximum rates using the proposed methods in Theorems 1 and 2, the main task is to find two 2×2 sub-matrices which satisfy certain properties. This is easier than finding eight 8×8 weighting matrices simultaneously as in the AOD approach [14].

Using Theorem 1 and Theorem 2, we construct here some *square* CO STBCs of order $N = 8$ (with or without LP) with the maximum number of variables $\mu(8) = 4$. The sub-matrices $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, $\mathbf{D}$ are of order $n = 2$ and each sub-matrix comprises one variable. From Theorem 1 (correspondingly, Theorem 2), it is clear that the most *crucial* task for constructing square CO STBCs of order $4n$ in our proposed methods is to find two matrices $\mathbf{A}$ and $\mathbf{B}$ ($\mathbf{A}$ and $\mathbf{C}$) satisfying the properties (2) and (3) in Theorem 1 (Theorem 2). We realize that various matrices $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, and $\mathbf{D}$ can satisfy those conditions, and derive here some of those cases for illustration.

Example 1: The following sub-matrices satisfy Theorem 1

$$\begin{aligned}\mathbf{A} &= k_1\begin{bmatrix}s_1 & s_1\\ -s_1^* & s_1^*\end{bmatrix}; \mathbf{B} = k_2\begin{bmatrix}-s_2^* & s_2^*\\ s_2 & s_2\end{bmatrix};\\ \mathbf{C} &= k_3\begin{bmatrix}-s_3^* & s_3^*\\ s_3 & s_3\end{bmatrix}; \mathbf{D} = k_4\begin{bmatrix}-s_4^* & s_4^*\\ s_4 & s_4\end{bmatrix},\end{aligned}$$

for any real coefficients k_i, $(i = 1,\ldots,4)$.

In this example, $\mathbf{A}$ is a variation of the Alamouti code with only one variable, while $\mathbf{C}$ and $\mathbf{D}$ are each of similar form to $\mathbf{B}$. Then, $\mathcal{O}$ in (16) satisfies $\mathcal{O}^H\mathcal{O} = 2\sum_{i=1}^4 k_i^2|s_i|^2\mathbf{I}_8$ and, consequently, $\mathcal{O}$ is a maximum rate, square, order-8 CO STBC (with or without LP depending on k_i). If $k_i = 1$, for $i = 1,\ldots,4$, from (16), we have the following code

$$\begin{bmatrix} s_1 & s_1 & -s_2^* & s_2^* & -s_3^* & s_3^* & -s_4^* & s_4^*\\ -s_1^* & s_1^* & s_2 & s_2 & s_3 & s_3 & s_4 & s_4\\ s_2^* & -s_2^* & s_1 & s_1 & -s_4 & s_4 & s_3 & -s_3\\ -s_2 & -s_2 & -s_1^* & s_1^* & s_4^* & s_4^* & -s_3^* & -s_3^*\\ s_3^* & -s_3^* & s_4 & -s_4 & s_1 & s_1 & -s_2 & s_2\\ -s_3 & -s_3 & -s_4^* & -s_4^* & -s_1^* & s_1^* & s_2^* & s_2^*\\ s_4^* & -s_4^* & -s_3 & s_3 & s_2 & -s_2 & s_1 & s_1\\ -s_4 & -s_4 & s_3^* & s_3^* & -s_2^* & -s_2^* & -s_1^* & s_1^*\end{bmatrix}. \tag{37}$$

Examples with various other structures are given below.

Example 2: This example illustrates the case in Theorem 1 where $\mathbf{C}$ and $\mathbf{D}$ are each of similar form to $\mathbf{B}^*$

$$\begin{aligned} \mathbf{A} &= k_1 \begin{bmatrix} s_1 & -s_1 \\ s_1^* & s_1^* \end{bmatrix}; \mathbf{B} = k_2 \begin{bmatrix} s_2^* & s_2^* \\ s_2 & -s_2 \end{bmatrix}; \\ \mathbf{C} &= k_3 \begin{bmatrix} s_3 & s_3 \\ s_3^* & -s_3^* \end{bmatrix}; \mathbf{D} = k_4 \begin{bmatrix} s_4 & s_4 \\ s_4^* & -s_4^* \end{bmatrix}. \end{aligned}$$

If $k_1 = k_2 = 1$ and $k_3 = k_4 = -1$, for instance, then we have

$$\begin{bmatrix} s_1 & -s_1 & s_2^* & s_2^* & -s_3 & -s_3 & -s_4 & -s_4 \\ s_1^* & s_1^* & s_2 & -s_2 & -s_3^* & s_3^* & -s_4^* & s_4^* \\ -s_2^* & -s_2^* & s_1 & -s_1 & -s_4^* & -s_4^* & s_3^* & s_3^* \\ -s_2 & s_2 & s_1^* & s_1^* & -s_4 & s_4 & s_3 & -s_3 \\ s_3 & s_3 & s_4^* & s_4^* & s_1 & -s_1 & s_2 & s_2 \\ s_3^* & -s_3^* & s_4 & -s_4 & s_1^* & s_1^* & s_2^* & -s_2^* \\ s_4 & s_4 & -s_3^* & -s_3^* & -s_2 & -s_2 & s_1 & -s_1 \\ s_4^* & -s_4^* & -s_3 & s_3 & -s_2^* & s_2^* & s_1^* & s_1^* \end{bmatrix}. \quad (38)$$

Example 3: This example using Theorem 1 shows that the CO STBC $\mathbf{G}8$ in (7) can be (indirectly) derived from our proposed methods. Let

$$\begin{aligned} \mathbf{A} &= k_1 \begin{bmatrix} s_1^* & s_1^* \\ s_1 & -s_1 \end{bmatrix}; \mathbf{B} = k_2 \begin{bmatrix} s_2 & -s_2 \\ s_2^* & s_2^* \end{bmatrix}; \\ \mathbf{C} &= k_3 \begin{bmatrix} s_3 & -s_3 \\ s_3^* & s_3^* \end{bmatrix}; \mathbf{D} = k_4 \begin{bmatrix} s_4 & -s_4 \\ s_4^* & s_4^* \end{bmatrix}. \end{aligned}$$

If $k_i = 1$ for $i = 1, \ldots, 4$, from (16), we have the following code

$$\begin{bmatrix} s_1^* & s_1^* & s_2 & -s_2 & s_3 & -s_3 & s_4 & -s_4 \\ s_1 & -s_1 & s_2^* & s_2^* & s_3^* & s_3^* & s_4^* & s_4^* \\ -s_2 & s_2 & s_1^* & s_1^* & s_4^* & -s_4^* & -s_3^* & s_3^* \\ -s_2^* & -s_2^* & s_1 & -s_1 & s_4 & s_4 & -s_3 & -s_3 \\ -s_3 & s_3 & -s_4^* & s_4^* & s_1^* & s_1^* & s_2^* & -s_2^* \\ -s_3^* & -s_3^* & -s_4 & -s_4 & s_1 & -s_1 & s_2 & s_2 \\ -s_4 & s_4 & s_3^* & -s_3^* & -s_2^* & s_2^* & s_1^* & s_1^* \\ -s_4^* & -s_4^* & s_3 & s_3 & -s_2 & -s_2 & s_1 & -s_1 \end{bmatrix}. \quad (39)$$

We note that the CO STBC $\mathbf{G}8$ in (7) can be derived from our CO STBC in (39) by multiplying every even row in (39) with j. However, $\mathbf{G}8$ in (7) itself does not follow our proposed structure as the sub-matrices $\mathbf{A}$ and $\mathbf{B}$ in $\mathbf{G}8$ do not satisfy the second condition in Theorem 1.

Example 4: This example illustrates the case in Theorem 2 where $\mathbf{B}$ and $\mathbf{D}$ are each of similar form to $\mathbf{A}$

$$\begin{aligned} \mathbf{A} &= k_1 \begin{bmatrix} s_1^* & s_1^* \\ s_1 & -s_1 \end{bmatrix}; \mathbf{B} = k_2 \begin{bmatrix} s_2^* & s_2^* \\ s_2 & -s_2 \end{bmatrix}; \\ \mathbf{C} &= k_3 \begin{bmatrix} s_3 & -s_3 \\ s_3^* & s_3^* \end{bmatrix}; \mathbf{D} = k_4 \begin{bmatrix} s_4^* & s_4^* \\ s_4 & -s_4 \end{bmatrix}. \end{aligned}$$

If $k_i = 1$ for $i = 1, \ldots, 4$, from (36), we have the following code

$$\begin{bmatrix} s_1^* & s_1^* & s_2^* & s_2^* & s_3 & -s_3 & s_4^* & s_4^* \\ s_1 & -s_1 & s_2 & -s_2 & s_3^* & s_3^* & s_4 & -s_4 \\ -s_2 & -s_2 & s_1 & s_1 & -s_4^* & -s_4^* & s_3 & -s_3 \\ -s_2^* & s_2^* & s_1^* & -s_1^* & -s_4 & s_4 & s_3^* & s_3^* \\ -s_3 & s_3 & s_4 & s_4 & s_1^* & s_1^* & -s_2 & -s_2 \\ -s_3^* & -s_3^* & s_4^* & -s_4^* & s_1 & -s_1 & -s_2^* & s_2^* \\ -s_4 & -s_4 & -s_3 & s_3 & s_2^* & s_2^* & s_1 & s_1 \\ -s_4^* & s_4^* & -s_3^* & -s_3^* & s_2 & -s_2 & s_1^* & -s_1^* \end{bmatrix}. \quad (40)$$

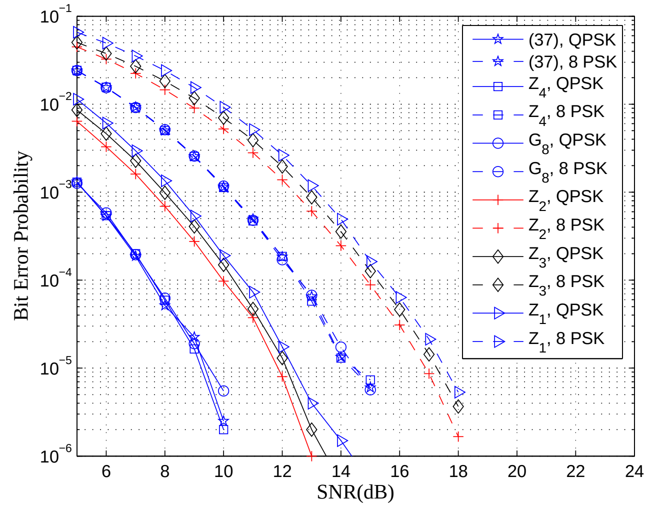

Fig. 1. The performance of the proposed code in (37), compared to the conventional code $\mathbf{Z}_1$, the codes $\mathbf{Z}_2$, $\mathbf{Z}_3$, $\mathbf{Z}_4$, and C. Yuen et al's code $\mathbf{G_8}$.

All of the above codes are square, maximum rate CO STBCs of order $N = 8$ with a full design, i.e., without any zeros for any complex signal constellations. The power is equally transmitted via each Tx antenna during every STS. For these reasons, the proposed CO STBCs are referred to as the *improved, square CO STBCs*.

V. Simulation Results

To examine the error performance of the proposed codes, we ran Monte-Carlo simulations for the code in (37) in a system with 8 Tx antennas and 1 receive (Rx) antenna for illustration. The bit error performance of the proposed code was analyzed in both QPSK and 8 PSK modulation schemes and was considered in a flat Rayleigh fading channel. The channel coefficients and noise are assumed to be i.i.d., zero-mean, complex Gaussian random variables. The SNR examined here is the channel SNR, i.e., the ratio between the sum of the average power of all received signals during a STS at the Rx antenna and the average noise power. The error performance of the conventional code $\mathbf{Z}_1$ in (1), the codes $\mathbf{Z}_2$ in (4), $\mathbf{Z}_3$ in (5), where several zero entries are contained in the code matrix, the code $\mathbf{Z}_4$ without zero entries mentioned in (6), and C. Yuen et al's code $\mathbf{G}_8$ in (7) were also shown in both QPSK and 8 PSK modulation schemes as the references. The Monte-Carlo simulations were run for 1,000,000 trials.

It is noted that the power of symbols transmitted through each Tx antenna in each STS was normalized to one in both QPSK and 8 PSK modulation cases for all considered codes. In particular, for the CO STBC in (37), the conventional code $\mathbf{Z}_1$, the codes $\mathbf{Z}_4$ and $\mathbf{G}_8$, all the transmitted symbols were derived from a unitary signal constellation. In $\mathbf{Z}_2$, the transmitted symbols s_1 and s_2 were derived from a unitary signal constellation, while the power of s_3 or s_4 was twice the power of s_1 or s_2. Similarly, for $\mathbf{Z}_3$, the transmitted symbols s_1, s_2 and s_3 were derived from a unitary signal constellation, while the power of s_4 was four times as much as that of s_1, s_2 or s_3.

By doing this, we stuck to the aim of transmitting the power of information-bearing symbols equally through each Tx antenna per STS, which is, in turn, one of the main purposes of this paper. In other words, we conditioned that the peak power per channel use was unitary and was the same for all considered codes in the simulations. Thus the average transmission power of the code (37), $\mathbf{Z}_4$, and $\mathbf{G}_8$ was 1, while that was 1/2 for $\mathbf{Z}_1$, 3/4 for $\mathbf{Z}_2$ and 7/8 for $\mathbf{Z}_3$, respectively. Equivalently, the PAPR of the proposed code and of $\mathbf{Z}_4$ and $\mathbf{G}_8$ was one, while that of $\mathbf{Z}_1$, $\mathbf{Z}_2$ and $\mathbf{Z}_3$ was 2, 4/3 and 8/7, respectively. We can see that having zeros in the code matrix results in a higher PAPR in comparison with the code with no zeros. Clearly, the average transmission power in the whole block of the code in (37) was twice as much as that in $\mathbf{Z}_1$ and equal to that in $\mathbf{Z}_4$ and in $\mathbf{G}_8$. Therefore, the simulation results are expected to show that the performance of the proposed code is 3dB better than that of $\mathbf{Z}_1$ and the same as that of $\mathbf{Z}_4$ and of $\mathbf{G}_8$. These observations have been confirmed in Fig. 1, where the proposed code provides approximately 3dB better bit error performance than $\mathbf{Z}_1$ at $BER = 10^{-4}$ in both QPSK and 8 PSK modulation schemes, while it provides the same bit error performance as $\mathbf{Z}_4$ and $\mathbf{G}_8$.

It is interesting to note that the overall error performance of the CO STBCs does not only depend on the average transmission power per symbol, but also depends on the structure of the codes. In particular, from the transmission power point of view, the gains of 1.25 dB (i.e. $10\lg(4/3)$) and of 0.58 dB (i.e. $10\lg(8/7)$) are theoretically expected to achieve by the code in (37) (also by $\mathbf{Z}_4$ or by $\mathbf{G}_8$) in comparison with $\mathbf{Z}_2$ and $\mathbf{Z}_3$, respectively. However, from Fig. 1, it can be realized that the code in (37) provides approximate 2.5 dB and 2.75 dB better error performances than $\mathbf{Z}_2$ and $\mathbf{Z}_3$, respectively, in both QPSK and 8 PSK modulation schemes. It can also be realized that $\mathbf{Z}_2$ actually provides better error performance than $\mathbf{Z}_3$, although the average transmission power per symbol in the whole block of the former is slightly smaller than the latter.

This observation can be explained as follows. $\mathbf{Z}_2$ provides more diversity in both spatial and temporal directions for the 4 bits embedded in the two symbols s_3 and s_4 in the case of QPSK modulation (6 bits in the case of 8 PSK modulation), while $\mathbf{Z}_3$ only provides more diversity for the 2 bits embedded in the symbol s_4 (3 bits in the 8 PSK modulation). Therefore, $\mathbf{Z}_2$ may provide a better resistance to burst errors than $\mathbf{Z}_3$. Similarly, the code in (37) provides more diversity in both spatial and temporal directions for the 8 bits embedded in the four symbols s_1, s_2, s_3 and s_4 in the case of QPSK modulation (12 bits in the case of 8 PSK modulation). In other words, the dispersion of symbols within the CO STBCs can be an important factor to result in a good bit error performance and it should be considered in designing a good CO STBC, besides the rank and determinant (or coding advantage) criteria [6], [17], [18]. From the mathematical viewpoint, the good dispersion means that there are as fewer zeros in the whole matrix as possible and that the non-zero entries are as much scattered in the whole matrix as possible.

VI. Conclusion

By modifying the Williamson and Wallis-Whiteman arrays to apply to complex matrices, we have proposed two new methods of constructing square, order-$4n$ CO STBCs from square, order-n CO STBCs which satisfy certain properties as described in Theorems 1 and 2. Applying Theorems 1 and 2, we have constructed various square, maximum rate, order-8 CO STBCs with no zeros. In our CO STBCs, the transmitted symbols equally disperse through Tx antennas with the consequence that the power can be equally transmitted via each Tx antenna during every STS. Additionally, it is our conjecture that the proposed methods can be applied to design square CO STBCs of order 16 or 32 from square CO STBCs of order 4 or 8, respectively, provided that there exist sub-matrices satisfying the conditions of our theorems. The construction of square CO STBCs of higher orders, such as 16 or 32, requires further study, and this is our future work.

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