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# Weighing Matrices and String Sorting

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**Abstract.** In this paper we establish a fundamental link between the search for weighing matrices constructed from two circulants and the operation of sorting strings, an operation that has been studied extensively in computer science. In particular, we demonstrate that the search for weighing matrices constructed from two circulants using the power spectral density criterion and exploiting structural patterns for the locations of the zeros in candidate solutions, can be viewed as a string sorting problem together with a linear time algorithm to locate common strings in two sorted arrays. This allows us to bring into bear efficient algorithms from the string sorting literature. We also state and prove some new enhancements to the power spectral density criterion, that allow us to treat successfully the rounding error effect and speed up the algorithm. Finally, we use these ideas to find new weighing matrices of order $2n$ and weights $2n-13$, $2n-17$ constructed from two circulants.

*Keywords*: weighing matrices, algorithm, pattern, locations of zeros, power spectral density, rounding error

## 1. Introduction

A weighing matrix $W = W(n, k)$ is a square matrix with entries $0, \pm 1$ having $k$ non-zero entries per row and column and inner product of distinct rows equal to zero. Therefore $W$ satisfies $WW^t = kI_n$. The number $k$ is called the weight of $W$. Weighing matrices have been studied extensively, see [8] and references therein. Weighing matrices are important in Coding Theory, for instance, they can be used [1] to construct

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self-dual codes. A well-known necessary condition for the existence of $W(2n,k)$ matrices states that if there exists a $W(2n,k)$ matrix with $n$ odd, then $k < 2n$ and $k$ is the sum of two squares. The two circulants construction for weighing matrices is described in the theorem below, taken from [6].

**Theorem 1.1.** *If there exist two circulant matrices $A$, $B$ of order $n$, with elements $0, \pm 1$, satisfying $AA^t + BB^t = kI_n$ and $k$ is an integer, then there exists a $W(2n,k)$, given as*

$$W(2n,k) = \begin{pmatrix} A & B \\ -B^t & A^t \end{pmatrix} \text{ or } W(2n,k) = \begin{pmatrix} A & BR \\ -BR & A \end{pmatrix},$$

*where $R$ is the square matrix of order $n$ with $r_{ij} = 1$ if $i + j - 1 = n$ and $0$ otherwise.*

## 2. Structural Patterns for the Location of the Zeros

We are interested in $W(2n, 2n-\alpha)$ weighing matrices constructed from two circulants, where $\alpha = 13, 17$. The weight $2n - \alpha$ implies that there are $\alpha$ zeros in total, in every row (and column) of such a matrix. Since we are using the two circulant matrices $A$ and $B$ as detailed in theorem 1.1, this further implies that if we denote by the arrays $[a_1, \ldots, a_n]$ and $[b_1, \ldots, b_n]$ the first rows of $A$ and $B$ in a potential solution, then these $\alpha$ zeros will be distributed in these two arrays.

The computational complexity of the exhaustive search for $W(2n, 2n-\alpha)$ weighing matrices constructed from two circulants is exponential. More precisely, a naive brute force exhaustive search would require $3^{2n} \approx 2^{3.17n}$ steps, which is already beyond the scope of existing algorithms for $n \geq 23$.

One way to reduce the computational complexity of the search for $W(2n, 2n-\alpha)$ weighing matrices constructed from two circulants is to exploit structural patterns for the locations of the $\alpha$ zeros in the two arrays $[a_1, \ldots, a_n]$ and $[b_1, \ldots, b_n]$. In this context, a structural pattern is a statement of the form

*there are $p$ zeros in $[a_1, \ldots, a_n]$ and $\alpha - p$ zeros in $[b_1, \ldots, b_n]$.*

To answer the question of what would be a reasonable value of $p$, we wrote a bash shell script metaprogram to generate via the Maple *CodeGeneration* package the C programs to perform exhaustive searches for $W(2n, 2n-\alpha)$ weighing matrices constructed from two circulants whose first rows are given by $a_1, \ldots, a_n$ and $b_1, \ldots, b_n$, and that follows the structural pattern

$$\underbrace{a_1 a_2 \ldots a_{n-1} a_n}_{p \text{ zeros}} \; \underbrace{b_1 b_2 \ldots b_{n-1} b_n}_{\alpha - p \text{ zeros}} \quad (p, \alpha - p) \text{ structural pattern}, \tag{2.1}$$

for $p \in \left\{0, 1, \ldots, \frac{\alpha+1}{2}\right\}$.

The results of these computations suggest that $p = \frac{\alpha+1}{2}$, which corresponds to the structural pattern $\left(\frac{\alpha+1}{2}, \frac{\alpha-1}{2}\right)$, is a reasonable value of $p$ to use, in order to tackle successfully the following open problems, see [8],

$$W(2 \cdot 27, 41), W(2 \cdot 29, 45), W(2 \cdot 31, 49),$$

$$W(2\cdot 33,53), W(2\cdot 37,61), W(2\cdot 39,65), \quad \text{for } \alpha = 13,$$
$$W(2\cdot 27,37), W(2\cdot 29,41), W(2\cdot 31,45),$$
$$W(2\cdot 33,49), W(2\cdot 35,53), W(2\cdot 39,61), \quad \text{for } \alpha = 17.$$

Using a $(p, \alpha - p)$ structural pattern for a specific value of $p$, the computational complexity of the search for $W(2n, 2n-\alpha)$ weighing matrices constructed from two circulants is reduced as follows:

**Lemma 2.1.** *An exhaustive search for $W(2n, 2n-\alpha)$ weighing matrices constructed from two circulants with a $(p, \alpha-p)$ structural pattern requires $\binom{n}{p}\binom{n}{\alpha-p}2^{2n-\alpha}$ steps.*

*Proof.* The $p$ zeros can be placed in the array $[a_1, \ldots, a_n]$ in $\binom{n}{p}$ different ways and the remaining $n-p$ $(\pm 1)$-elements can be chosen in $2^{n-p}$ different ways. The $\alpha - p$ zeros can be placed in the array $[b_1, \ldots, b_n]$ in $\binom{n}{\alpha-p}$ different ways and the remaining $n-(\alpha-p)$ $(\pm 1)$-elements can be chosen in $2^{n-(\alpha-p)}$ different ways. Consequently, the exhaustive search for $W(2n, 2n-\alpha)$ weighing matrices constructed from two circulants with a $(p, \alpha-p)$ structural pattern requires $\binom{n}{p}2^{n-p}\binom{n}{\alpha-p}2^{n-(\alpha-p)} = \binom{n}{p}\binom{n}{\alpha-p}2^{2n-\alpha}$ steps. ∎

Setting $p = \frac{\alpha+1}{2}$ in the previous lemma, we obtain

**Corollary 2.2.** *An exhaustive search for $W(2n, 2n-\alpha)$ weighing matrices constructed from two circulants with an $\left(\frac{\alpha+1}{2}, \frac{\alpha-1}{2}\right)$ structural pattern requires $\binom{n}{\frac{\alpha+1}{2}}\binom{n}{\frac{\alpha-1}{2}}2^{2n-\alpha}$ steps.*

The paper [8] contains a number of relevant conjectures on weighing matrices as well as a comprehensive list of open cases for $W(2n, k)$ constructed from two circulants. Here we focus on $W(2n, 2n-\alpha)$ weighing matrices constructed from two circulants, for $\alpha = 13, 17$. Results on the structure of weighing matrices $W(n, n-2)$, $W(n, n-3)$, $W(n, n-4)$ (not necessarily constructed from two circulants) are given in [3].

## 3. The Power Spectral Density Criterion for Weighing Matrices

It is well known that if the Diophantine equation $a^2 + b^2 = k$ has no solutions, then there do not exist $W(2n, k)$ weighting matrices constructed from two circulants, and therefore we focus our attention on the permissible odd values of $n$, i.e., values of $n$ such that the Diophantine equation $a^2 + b^2 = 2n - \alpha$ has solutions. It is also well known that the first rows $[a_1, \ldots, a_n]$, $[b_1, \ldots, b_n]$ of the circulant matrices $A$ and $B$ that make up $W(2n, k)$ weighing matrices, satisfy the Diophantine equation $a^2 + b^2 = k$ where $a$, $b$ are the row sums, i.e., $a = a_1 + \cdots + a_n$, $b = b_1 + \cdots + b_n$. In addition, the power spectral density criterion [5] can be used to search efficiently for weighing matrices. Indeed, for odd $n$, two sequences $[a_1, \ldots, a_n]$, $[b_1, \ldots, b_n]$ can be used to make up circulant matrices $A$ and $B$ that will give $W(2n, k)$ weighing matrices if and only if

$$PSD([a_1, \ldots, a_n], i) + PSD([b_1, \ldots, b_n], i) = k, \quad \forall i = 0, \ldots, \frac{n-1}{2}, \tag{3.1}$$

where $PSD([a_1,\dots,a_n],i)$ denotes the $i$-th element of the power spectral density sequence, i.e., the square magnitude of the $i$-th element of the discrete Fourier transform sequence associated to $[a_1,\dots,a_n]$ which is defined as $DFT_{[a_1,\dots,a_n]} = [\mu_0,\dots,\mu_{n-1}]$, with $\mu_j = \sum_{i=0}^{n-1} a_{i+1}\,\omega^{ij}$, $j=0,\dots,n-1$, where $\omega = e^{\frac{2\pi i}{n}} = \cos\left(\frac{2\pi}{n}\right) + i\sin\left(\frac{2\pi}{n}\right)$ is a primitive $n$-th root of unity (also called the principal $n$-th root of unity). Keeping in mind that the elements of the power spectral density sequences are non-negative, the power spectral density criterion for $W(2n,k)$ weighing matrices can be stated as: If for a certain sequence $[a_1,\dots,a_n]$ there exists $i \in \{1,\dots,\frac{n-1}{2}\}$ with the property that $PSD([a_1,\dots,a_n],i) > k$, then this sequence can be discarded from the search. Note that the case $i=0$ has been accounted for via the Diophantine constraint.

### 3.1. Rounding Error Analysis for the Power Spectral Density Criterion

The elements of the DFT vector associated to a sequence are usually complex numbers with floating point real and imaginary parts. By implication, the elements of the corresponding PSD vector are usually floating point numbers. However, there are some cases in which some elements of the DFT vector can be evaluated in simple closed forms, i.e., their real and imaginary parts are rational numbers or algebraic numbers of small degree. By implication, the corresponding elements of the PSD vector are non-negative integers. This phenomenon becomes important in the search for $W(2n, 2n-\alpha)$ weighing matrices and must be taken into account, otherwise the rounding error effects will cause the algorithm to fail to find solutions, when solutions do actually exist. The following lemmas identify some cases in which one can observe the phenomenon described above.

**Lemma 3.1.** *Let $n$ be an odd integer such that $n \equiv 0 \pmod 3$ and let $m = \frac{n}{3}$. Let $\omega = e^{\frac{2\pi i}{n}} = \cos\left(\frac{2\pi}{n}\right) + i\sin\left(\frac{2\pi}{n}\right)$ be the principal $n$-th root of unity. Let $[a_1,\dots,a_n]$ be a sequence with elements from $\{-1,0,+1\}$. Then we have that $DFT([a_1,\dots,a_n],m)$ and $PSD([a_1,\dots,a_n],m)$ can be evaluated explicitly in closed form and moreover, $PSD([a_1,\dots,a_n],m)$ is a non-negative integer. The explicit evaluations are given by*

$$DFT([a_1,\dots,a_n],m) = \left(A_1 - \frac{1}{2}A_2 - \frac{1}{2}A_3\right) + \left(\frac{\sqrt{3}}{2}A_2 - \frac{\sqrt{3}}{2}A_3\right)i,$$

$$PSD([a_1,\dots,a_n],m) = A_1^2 + A_2^2 + A_3^2 - A_1A_2 - A_1A_3 - A_2A_3,$$

*where*

$$A_1 = \sum_{i=0}^{m-1} a_{3i+1}, \quad A_2 = \sum_{i=0}^{m-1} a_{3i+2}, \quad A_3 = \sum_{i=0}^{m-1} a_{3i+3}.$$

*Proof.* We remark that $DFT([a_1,\dots,a_n],m)$ is a linear combination of $\omega^0, \omega^m, \omega^{2m}$, more specifically, we have

$$DFT([a_1,\dots,a_n],m) = \sum_{i=0}^{n-1} a_{i+1}\,\omega^{im}$$

$$= \sum_{j=1}^{3} \left( \left( \sum_{i=0}^{m-1} a_{3i+j} \right) \omega^{(j-1)m} \right)$$

$$= A_1\omega^0 + A_2\omega^m + A_3\omega^{2m}.$$

Now we remark that $\omega^m = e^{\frac{2\pi i}{3}}$ and $\omega^{2m} = e^{\frac{4\pi i}{3}}$ which are the roots of the cyclotomic polynomial $\Phi_3(x) = x^2 + x + 1$ and can be evaluated explicitly as: $\omega^m = -\frac{1}{2} + \frac{\sqrt{3}}{2}i$, $\omega^{2m} = -\frac{1}{2} - \frac{\sqrt{3}}{2}i$. Therefore, we have for the real and imaginary parts of $DFT([a_1,\dots,a_n],m)$ that: $\Re(DFT([a_1,\dots,a_n],m)) = A_1 - \frac{1}{2}A_2 - \frac{1}{2}A_3$, $\Im(DFT([a_1,\dots,a_n],m)) = \frac{\sqrt{3}}{2}A_2 - \frac{\sqrt{3}}{2}A_3$, and the explicit evaluation of $PSD([a_1,\dots,a_n],m)$ and the fact that it is an integer follow. ∎

**Corollary 3.2.** *$PSD([a_1,\dots,a_n],m) = \frac{3}{2}\left(A_1^2+A_2^2+A_3^2\right) - \frac{a^2}{2}$ where $a$ is such that $a = a_1 + \cdots + a_n$.*

*Proof.* The Jacobi-Trudi formula states that $e_2 = \frac{1}{2!}\begin{vmatrix} p_1 & 1 \\ p_2 & p_1 \end{vmatrix} = \frac{p_1^2}{2} - \frac{p_2}{2}$ where $e_1$, $e_2$ are the first and second elementary symmetric functions and $p_1$, $p_2$ are the first and second power sums. We now regard $A_1$, $A_2$, $A_3$ as variables and by applying the first of the above identities to the explicit evaluation for $PSD([a_1,\dots,a_n],m)$ we obtain: $PSD([a_1,\dots,a_n],m) = A_1^2+A_2^2+A_3^2-A_1A_2-A_1A_3-A_2A_3 = p_2 - e_2 = p_2 - \left(\frac{p_1^2}{2} - \frac{p_2}{2}\right) = \frac{3}{2}p_2 - \frac{1}{2}p_1^2$. Noting that $p_1 = A_1 + A_2 + A_3 = \sum_{i=1}^{n} a_i = a$ completes the proof. ∎

*Remark 3.3.* The fact that $PSD([a_1,\dots,a_n],m)$ turns out to be a symmetric polynomial in $A_1, A_2, A_3$ will most probably play a role in the determination of the range of $PSD([a_1,\dots,a_n],m)$ values over the entire search space of sequences. The experimental results so far suggest that the range of all possible values of $PSD([a_1,\dots,a_n],m)$ is usually a very small set.

**Lemma 3.4.** *Let $n$ be an odd integer such that $n \equiv 0 \,(mod\, 5)$ and let $m = \frac{n}{5}$. Let $\omega = e^{\frac{2\pi i}{n}} = \cos\left(\frac{2\pi}{n}\right) + i\sin\left(\frac{2\pi}{n}\right)$ be the principal $n$-th root of unity. Let $[a_1,\dots,a_n]$ be a sequence with elements from $\{-1,0,+1\}$. Then we have that $DFT([a_1,\dots,a_n],m)$ and $PSD([a_1,\dots,a_n],m)$ can be evaluated explicitly in closed form. The explicit evaluations are given by*

$$DFT([a_1,\dots,a_n],m) = \left(A_1 + \frac{\sqrt{5}-1}{4}(A_2+A_5) - \frac{\sqrt{5}+1}{4}(A_3+A_4)\right)$$

$$+ \left(\frac{\sqrt{10+2\sqrt{5}}}{4}(A_2-A_5) + \frac{\sqrt{10-2\sqrt{5}}}{4}(A_3-A_4)\right)i,$$

$$PSD([a_1,\dots,a_n],m) = A_1^2+A_2^2+A_3^2+A_4^2+A_5^2$$

$$- \frac{1}{2}(A_1A_2 + A_1A_3 + A_1A_4 + A_1A_5 + A_2A_3 + A_2A_4 + A_2A_5 + A_3A_4 + A_3A_5 + A_4A_5) + \frac{\sqrt{5}}{2}(A_1A_2 - A_1A_3 - A_1A_4 + A_1A_5 + A_2A_3 - A_2A_4 - A_2A_5 + A_3A_4 - A_3A_5 + A_4A_5),$$

*where*

$$A_1 = \sum_{i=0}^{m-1} a_{5i+1},\ A_2 = \sum_{i=0}^{m-1} a_{5i+2},\ A_3 = \sum_{i=0}^{m-1} a_{5i+3},\ A_4 = \sum_{i=0}^{m-1} a_{5i+4},\ A_5 = \sum_{i=0}^{m-1} a_{5i+5}.$$

The proof of lemma 3.4 is similar to the proof of lemma 3.1 and is based on the facts that $DFT([a_1,\ldots,a_n],m)$ is a linear combination of $\omega^0$, $\omega^m$, $\omega^{2m}$, $\omega^{3m}$, $\omega^{4m}$ and that $\omega^m$, $\omega^{2m}$, $\omega^{3m}$, $\omega^{4m}$ are the roots of the cyclotomic polynomial $\Phi_5(x) = x^4 + x^3 + x^2 + x + 1$ that can be evaluated explicitly as

$$\frac{\sqrt{5}-1}{4} + \frac{\sqrt{10+2\sqrt{5}}}{4}i, \qquad -\frac{\sqrt{5}+1}{4} + \frac{\sqrt{10-2\sqrt{5}}}{4}i,$$
$$-\frac{\sqrt{5}+1}{4} - \frac{\sqrt{10-2\sqrt{5}}}{4}i, \qquad \frac{\sqrt{5}-1}{4} - \frac{\sqrt{10+2\sqrt{5}}}{4}i.$$

## 4. String Sorting, Description of the Algorithm

We now describe how the search for weighing matrices can be thought of as a string sorting problem and then we give a detailed description of the algorithm. We rewrite Equation (3.1) in the form

$$PSD([a_1,\ldots,a_n],i) = k - PSD([b_1,\ldots,b_n],i), \quad \forall\, i = 0,\ldots,\frac{n-1}{2}, \tag{4.1}$$

and by taking integer parts and keeping in mind that $k$ is an integer, we obtain, for an arbitrary $i$,

$$\lfloor PSD([a_1,\ldots,a_n],i)\rfloor = \begin{cases} k - 1 - \lfloor PSD([b_1,\ldots,b_n],i)\rfloor, & \text{if } PSD([b_1,\ldots,b_n],i) \text{ is not an integer,} \\ k - \lfloor PSD([b_1,\ldots,b_n],i)\rfloor, & \text{if } PSD([b_1,\ldots,b_n],i) \text{ is an integer.} \end{cases} \tag{4.2}$$

Note that (3.1) implies that when $PSD([a_1,\ldots,a_n],i)$ is an integer, $PSD([b_1,\ldots,b_n],i)$ is also an integer. In the sequel, we denote $q = \frac{n-1}{2}$. We remark that for $i = 0$ the condition (3.1) is the same as the Diophantine constraint $a^2 + b^2 = k$ and that is why we exclude the index $i = 0$ from the encoding that we will describe below. In view of the previous discussion, a pair of vectors $[a_1, \ldots, a_n]$ and $[b_1, \ldots, b_n]$ can be encoded as the concatenation of the integer parts of the first $q$ components of their PSD vectors, using the relation (4.2). Using this encoding, the condition that a pair of vectors can be used as the first rows of circulants to construct $W(2n, 2n-\alpha)$ weighing matrices via theorem 1.1, can be simply phrased by saying that their corresponding string encodings are equal. Note that string equality does not imply (3.1). Therefore, we see that the search for weighing matrices is essentially a string sorting problem. For some sequences of length $n$, we may still have that an element of the PSD vector other than the $n/3$ or $n/5$ is an integer. In this case, the string encoding is not suitable, therefore, this algorithm is not an exhaustive search. We now give a high-level description of the algorithm, taking into consideration all the issues analyzed previously. This high-level description uses black box algorithms for generating permutations and for string sorting. This allows us to plug in efficient algorithms for these tasks to our implementation. See [9] for efficient algorithms for generating permutations and [7] for efficient algorithms for string sorting.

*Input*: An integer $\alpha$, an odd permissible value of $n$ and $p \in \left\{0, 1, \ldots, \frac{\alpha+1}{2}\right\}$.
*Output*: Pairs of vectors $[a_1, \ldots, a_n]$ and $[b_1, \ldots, b_n]$ following the $(p, \alpha-p)$ structural pattern, that can be used as the first rows of circulant matrices $A$, $B$ to construct $W(2n, 2n-\alpha)$ weighing matrices.
*Step 1*: Fix a solution $(a_f, b_f)$ of the Diophantine constraint $a^2+b^2 = 2n-\alpha$, taking into account the parity of $p$. Hard-code the computation of the DFT vector, essentially hard-code a Fast Fourier Transform (FFT) algorithm. Set $q = (n-1)/2$.
*Step 2*: Generate all vectors $[a_1, \ldots, a_n]$, $[b_1, \ldots, b_n]$ such that $a_i, b_i \in \{-1, 0, +1\}$, $\mid a_1 + \cdots + a_n \mid = a_f$, $\mid b_1 + \cdots + b_n \mid = b_f$, exactly $p$ of the $a_i$'s are equal to 0 and exactly $\alpha - p$ of the $b_i$'s are equal to 0, $PSD([a_1, \ldots, a_n], i) \leq k$, $\forall\, i = 1, \ldots, q$, $PSD([b_1, \ldots, b_n], i) \leq k$, $\forall\, i = 1, \ldots, q$.
If $n \equiv 0(\mod 3)$ then set $m = \frac{n}{3}$. Encode $[b_1, \ldots, b_n]$ by $\lfloor PSD([b_1, \ldots, b_n], 1)\rfloor \cdots \lfloor PSD([b_1, \ldots, b_n], q)\rfloor$.
If $n \not\equiv 0(\mod 3)$ then encode $[a_1, \ldots, a_n]$ by $k-1-\lfloor PSD([a_1, \ldots, a_n], 1)\rfloor \cdots k-1-\lfloor PSD([a_1, \ldots, a_n], q)\rfloor$.
If $n \equiv 0(\mod 3)$ then encode $[a_1, \ldots, a_n]$ by $k-1-\lfloor PSD([a_1, \ldots, a_n], 1)\rfloor \cdots k-\lfloor PSD([a_1, \ldots, a_n], m)\rfloor \cdots k-1-\lfloor PSD([a_1, \ldots, a_n], q)\rfloor$.
*Step 3*: If $n \equiv 0(\mathrm{mod}\ 3)$ or $n \equiv 0(\mathrm{mod}\ 5)$, then partition the sets of the string encodings of the $[a_1, \ldots, a_n]$ and $[b_1, \ldots, b_n]$ vectors, so that disjoint subsets contain vectors with equal $\lfloor PSD([a_1, \ldots, a_n], m)\rfloor$ and $\lfloor PSD([b_1, \ldots, b_n], m)\rfloor$ values.
*Step 4*: Sort the string encodings for all the $[a_1, \ldots, a_n]$ and $[b_1, \ldots, b_n]$ vectors, using the partition of the previous step, if applicable.
*Step 5*: Find the common strings between the two sorted sets of $[a_1, \ldots, a_n]$, $[b_1, \ldots, b_n]$ encodings. Recover the vectors corresponding to common strings and output those vectors that are actual solutions, i.e., that satisfy (3.1).

## 5. Results

We used the power spectral density criterion together with the rounding error analysis and string sorting, to search for $W(2n, 2n-13)$ for $n = 27, 29, 31, 33$ with the

$(7,6)$ structural pattern and $W(2n,2n-17)$ for $n = 27, 29, 31, 33, 35$ with the $(9,8)$ structural pattern. We found a number of new such weighing matrices, which are given here for the first time. Lists of known and unknown weighing matrices can be found in [2–4, 8]. The solutions are given in the format $a_1,\ldots,a_n, b_1,\ldots,b_n$, where these sequences are the first rows of the circulant matrices $A$ and $B$ in Theorem 1.1. Moreover, $-$ stands for $-1$ and $+$ stands for 1. The string encodings corresponding to the solutions are also given.

### 5.1. New Weighing Matrices $W(2n,2n-13)$

```
W(2*27,41) string encoding 101030264331635132630320
- - - - - - 0 - + + 0 - + + 0 0 0 - + - + 0 - - + 0 +
- - - - - 0 + - + 0 - + + - - 0 + 0 - + - 0 - - + 0 +

W(2*29,45) string encoding 1426252516253826429263425
- - - - - - - 0 + + - - 0 0 0 + - - 0 + + 0 - + - + - 0 +
- - - - - - 0 + + - + - 0 + 0 - + 0 - 0 + 0 - + - + - + +

W(2*31,49) string encoding 3827271911138119104224273644
- - - - - - 0 0 + 0 + - + 0 - 0 + + - + - + + 0 0 + - + - + +
- - - - - - 0 + - 0 + - - 0 + - 0 + - + - - - 0 + + 0 - - + +

W(2*33,53) string encoding 383333351462832942864313440
- - - - - - - 0 + - 0 + + - + - + - 0 + + 0 - 0 0 + + - 0 - + + +
- - - - - - 0 0 + 0 - + - 0 + - + + - - + - 0 - + - - + - - 0 + +
```

### 5.2. New Weighing Matrices $W(2n,2n-17)$

```
W(2*27,37) string encoding: 101330341726814253391219
- - - - - - 0 - + - 0 + 0 - - + 0 0 0 0 - + 0 + - 0 +
- - - - - + + 0 - + - + - - + 0 0 0 0 + + - 0 0 0 + +

W(2*29,41) string encoding: 18102326726302392117253218
- - - - - - 0 0 + - + 0 0 - - + + - + - - + + 0 0 0 0 0 +
- - - - - - 0 + 0 0 - - + + - 0 + - + - + - 0 0 0 0 - + +

W(2*31,45) string encoding: 1224251422176282237443793019
- - - - - - - - + + - 0 0 0 - 0 + 0 0 0 - + - + 0 + - - 0 + +
- - - - - 0 - + 0 + - + 0 - + 0 - + + - 0 - 0 + - + 0 - 0 + +

W(2*33,49)  string encoding: 101616396515322129133943413026
- - - - - - - + + - 0 + - + + 0 - 0 0 + 0 + 0 0 - + + 0 0 - + + +
- - - - - - 0 + 0 - - + 0 - 0 + - + 0 - 0 0 - + - + - - + - + 0 +

W(2*35,53)  string encoding: 2762304051530462140281432222927
- - - - - - - 0 0 + + - + - 0 0 + - + + - 0 - 0 + 0 + - 0 + 0 - + + +
- - - - - - 0 + - - 0 + - 0 - + - 0 0 - + 0 + 0 0 - - + + - + - - + +
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