On the pivot structure for the weighing matrix $W(12,11)$

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Abstract

C. Koukouvinos, M. Mitrouli and Jennifer Seberry, in "Growth in Gaussian elimination for weighing matrices, $W(n, n-1)$", *Linear Algebra and its Appl.*, 306 (2000), 189-202, conjectured that the growth factor for Gaussian elimination of any completely pivoted weighing matrix of order n and weight $n-1$ is $n-1$ and that the first and last few pivots are $(1, 2, 2, 3 \text{ or } 4, \ldots, n-1 \text{ or } \frac{n-1}{2}, \frac{n-1}{2}, n-1)$ for $n > 14$.

In the present paper we concentrate our study on the growth problem for the weighing matrix $W(12,11)$ and we show that the unique $W(12,11)$ has three pivot structures.

Key Words and Phrases: Gaussian elimination, growth, complete pivoting, weighing matrices.

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1 Introduction

Gaussian elimination (GE) is the simplest way to solve linear systems of equations by hand, and also the standard method for solving them on computers. The strategy of GE in order to solve the system $A \cdot \underline{x} = \underline{b}$, where $A = [a_{ij}] \in \mathcal{R}^{n \times n}$ is nonsingular, is to reduce the full linear system to a triangular system that can be easily solved, using elementary row operations. This can be achieved by applying simple linear transformations on the left. There are $n-1$ stages, beginning with $A^{(1)} := A$, $\underline{b}^{(1)} := \underline{b}$ and finishing with the upper triangular system $A^{(n)} \cdot \underline{x} = \underline{b}^{(n)}$. Let $A^{(k)} = [a_{ij}^{(k)}]$ denote the matrix obtained after the first k pivoting operations, so $A^{(n)}$ is the final upper triangular matrix. A diagonal entry of that final matrix will be called a pivot. Matrices with the property that no exchanges are actually needed during GE with complete pivoting are called completely pivoted (CP) or feasible.

Traditionally, backward error analysis for GE is expressed in terms of the *growth factor*

$$g(n, A) = \frac{max_{i,j,k}|a_{ij}^{(k)}|}{max_{i,j}|a_{ij}|},$$

which involves all the elements $a_{ij}^{(k)}$, $k = 1, 2, \ldots, n$ that occur during the elimination. For a CP matrix A we have

$$g(n, A) = \frac{max\{p_1, p_2, \ldots, p_n\}}{|a_{11}|},$$

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where $p_1, p_2, \ldots, p_n$ are the pivots of A. For GE with partial pivoting it is proved that $g(n, A) \leq 2^{n-1}$. The following theorem illustrates the accuracy of the computed solution.

Theorem 1 *(Wilkinson) Let $A =\in \mathcal{R}^{n\times n}$ and suppose GE with partial pivoting produces a computed solution $\hat{x}$ to $A \cdot x = b$. Then*

$$(A + \Delta A)\hat{x} = b, \quad \|\Delta A\|_\infty \leq cn^2 g(n, A)\|A\|_\infty.$$

It is obvious that the stability of GE depends on the growth factor. If $g(n, A)$ is of order 1, not much growth has taken place, and the elimination process is stable. If $g(n, A)$ is bigger than this, we must expect instability. An arising question is whether GE is backward stable. An algorithm is characterized as backward stable if it gives exactly the right answer to nearly the right question. According to this definition and the above theorem, the answer to the question is yes if $g(n, A)$ is of order 1 uniformly for all matrices of a given dimension n, otherwise no. GE with partial pivoting is backward stable. This conclusion is absurd, however, in view of the vastness of 2^{n-1} for practical values of n.

If GE is unstable, why is it so famous and so popular? Despite some examples of very specific matrices, GE with partial pivoting is utterly stable in practice. In fifty years of computing, no matrix problems that excite an explosive instability are known to have arisen under natural circumstances. But how can an algorithm that fails for certain matrices be entirely trustworthy in practice? The answer seems to be that although some matrices cause instability, these represent such an extraordinary small proportion of the set of all matrices that they "never" arise in practise simply for statistical reasons. This explanation gives rise to a statistical approach to the growth factor [**?**].

For a CP matrix A Cryer [**?**] defined

$$g(n) = \sup\{\, g(n, A)/A \in \mathcal{R}^{n\times n}\,\}.$$

The problem of determining $g(n)$ for various values of n is called the *growth problem.*

The following results are known:

- $g(2) = 2$ (trivial)
- $g(3) = 2\frac{1}{4}$ [**?**]
- $g(4) = 4$ [**?**],[**?**]
- $g(5) < 5.005$ [**?**]

The determination of $g(n)$ in general remains a mystery. Wilkinson in [**?**] proved that

$$g(n) \leq [n\, 2\, 3^{1/2} \ldots n^{1/(n-1)}]^{1/2} \sim cn^{1/2} n^{\frac{1}{4}\log n}$$

and that this bound is not attainable. The bound is a much more slowly growing function than 2^{n-1}, but it can still be quite large (e.g. it is 3570 for $n = 100$). As for partial pivoting, in practice the growth factor is usually small. Wilkinson in [**?**] and [**?**] noted that there were no known examples of matrices for which $g(n) > n$. In [**?**] Cryer conjectured that "$g(n, A) \leq n$, with equality iff A is a Hadamard matrix". This conjecture became one of the most famous open problems in numerical analysis and has been investigated by many mathematicians. In 1991 Gould [?] discovered a 13×13 matrix for which the growth factor is 13.0205. Thus the first part of the conjecture was shown to be false. The second part of the conjecture concerning the growth factor of Hadamard matrices still remains open. Interesting problems remain, such as determining $\lim_{n\to\infty} g(n)/n$ and evaluating $g(n, A)$ for Hadamard matrices.

An Hadamard matrix H of order $n \times n$ is an orthogonal matrix with elements ± 1 and $HH^T = nI$.

Two matrices are said to be *Hadamard equivalent* or *H-equivalent* if one can be obtained from the other by a sequence of the operations:

1. interchange any pairs of rows and/or columns;

2. multiply any rows and/or columns through by -1.

A $(0, 1, -1)$ matrix $W = W(n, k)$ of order n satisfying $WW^T = kI_n$ is called a *weighing matrix of order n and weight k* or simply a *weighing matrix.* A $W(n, n)$, $n \equiv 0 \pmod 4$, is a Hadamard matrix of order n. A $W = W(n, k)$ for which $W^T = -W$,$n \equiv 0 \pmod 4$, is called a *skew–weighing matrix.* A $W = W(n, n-1)$ satisfying $W^T = W$, $n \equiv 2 \pmod 4$, is called a *symmetric conference matrix.* Conference matrices cannot exist unless $n-1$ is the sum of two squares: thus they cannot exist for orders $22, 34, 58, 70, 78, 94$. For more details and construction of weighing matrices the reader can consult the book of Geramita and Seberry [?].

Wilkinson's initial conjecture seems to be connected with Hadamard matrices. Interesting results in the size of pivots appear when GE is applied to CP weighing matrices of order n and weight $n-1$. In [?] has been studied the growth problem for CP skew and symmetric conference matrices. In these matrices, the growth is also large, and experimentally, we have been led to believe it equals $n-1$ and special structure appears for the first few and last few pivots.

In the present paper we calculate theoretically the pivot structure of the $W(12, 11)$ by making use of specific techniques and algorithms. Finally we are led to prove that the $W(12, 11)$ has 3 pivot patterns, which was already stated only experimentally, and we are in position to state that the growth factor for $W(12, 11)$ is equal to 11. So there is proved the growth conjecture [?] for the skew-weighing matrix $W(12, 11)$.

Notation. Throughout this paper the elements of a $(0, 1, -1)$ matrix will be denoted by $(0, +, -)$. Write A for a matrix of order n whose initial pivots are derived from matrices with CP structure. Write $A(j)$ for the absolute value of the determinant of the $j \times j$ principal submatrix in the upper lefthand corner of the matrix A. It can be proved that the magnitude of the pivots appearing after the application of GE operations on a CP matrix W is given by

$$p_j = W(j)/W(j-1), \quad j = 1, 2, \ldots, n, \quad W(0) = 1. \tag{1}$$

We use $W(j)$ similarly.

2 Extension of specific $(0, +, -)$ matrices to $W(n, n-1)$

The first four pivots

Since pivots are strictly connected with minors we start our study with an effort of computing principal minors of skew and symmetric conference matrices. The following lemma specifies the possible values of determinants of small order. The results for order 8 are new.

Lemma 1 *The maximum absolute value of the determinant of all $n \times n$ $(0, +, -)$ matrices, where there is at most one zero in each row and column, is given in the following table for $n = 2, 3, 4, 5, 6, 7, 8$:*

Order	*Maximum Determinant*	*Possible Determinantal Values*
2×2	2	$0, 1, 2$
3×3	4	$0, 1, 2, 3, 4$
4×4	16	$0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 12, 16$
5×5	48	$0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18,$ $19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 30, 32, 36, 40, 48$
6×6	160	$160, 144, 136, 132, 130, 128, 120, 112, 108, 106, 105, 104,$ $102, 100, \ldots$
7×7	528	$528, 504, 480, 468, 456, 444, 440, 432, 420, 408, 396, 384, 372,$ $366, 360, 354, 348, 342, 336, 330, 324, \ldots$
8×8	2224	$2224, 2168, 2112, 2096, 2088, 2064, 2032, 2008,$ $1984, 1972, 1968, 1940, 1936, 1904, \ldots$

□

In [?] were proved the following lemmas 2,3 and 4:

Lemma 2 *Let W be a CP skew and symmetric matrix, of order $n \geq 6$ then if GE is performed on W the first two pivots are 1 and 2.*

Lemma 3 *Let W be a CP skew and symmetric conference matrix, of order $n \geq 12$ then if GE is performed on W the third pivot is 2.*

Lemma 4 *Let W be a CP skew and symmetric conference matrix, of order $n \geq 12$ then if GE is performed on W the fourth pivot is 3 or 4.*

Next we focus our study to $W(12, 11)$.

Lemma 5 *The following submatrices always occur in a $W(12, 11)$:*

$$E_1 = \begin{bmatrix} + & + & + & + & + & + & + \\ + & - & + & - & - & - & - \\ + & - & - & + & + & + & - \\ + & + & - & - & + & - & 0 \\ + & + & - & + & - & - & - \\ + & + & - & - & - & + & + \\ + & - & - & + & 0 & - & + \end{bmatrix}$$

$$E_2 = \begin{bmatrix} + & + & 0 & - & + & + & + \\ + & - & - & - & 0 & - & + \\ + & - & + & + & + & - & - \\ + & + & - & + & - & + & - \\ + & + & + & - & - & - & - \\ + & - & + & - & - & + & + \\ + & + & + & + & - & - & + \end{bmatrix}$$

$$E_3 = \begin{bmatrix} + & + & 0 & - & + & + & + \\ + & - & - & - & - & + & - \\ + & - & + & + & + & - & - \\ + & + & - & + & - & - & - \\ + & + & + & - & - & - & + \\ + & 0 & + & + & - & + & + \\ + & - & - & 0 & - & - & + \end{bmatrix}$$

Proof. We note that $W(12,11)$ is unique up to H-equivalence. Hence it is sufficient to demonstrate that $E_1,\ E_2$ *and* E_3 exist in $W(12,11)$.
Consider the following $W(12,11)$:

$$\begin{bmatrix} + & + & + & + & + & + & + & 0 & + & + & - & - \\ + & - & + & - & - & - & - & - & + & + & - & 0 \\ + & - & - & + & + & + & - & - & 0 & - & - & + \\ + & + & - & - & + & - & 0 & - & + & - & + & - \\ + & + & - & + & - & - & - & + & - & 0 & - & - \\ + & + & - & - & - & + & + & - & - & + & 0 & + \\ + & - & - & + & 0 & - & + & + & + & + & + & + \\ 0 & + & + & + & + & - & - & - & - & + & + & + \\ - & + & 0 & + & - & - & + & - & + & - & - & + \\ - & - & - & 0 & + & - & + & - & - & + & - & - \\ + & - & + & + & - & 0 & + & - & - & - & + & - \\ - & 0 & - & + & - & + & - & - & + & + & + & - \end{bmatrix}$$

$$\begin{bmatrix} + & + & 0 & - & + & + & + & + & + & - & + & - \\ + & - & - & - & 0 & - & + & + & + & + & - & + \\ + & - & + & + & + & - & - & 0 & + & - & - & - \\ + & + & - & + & - & + & - & + & 0 & - & - & + \\ + & + & + & - & - & - & - & - & + & 0 & + & + \\ + & - & + & - & - & + & + & - & - & - & - & 0 \\ + & + & + & + & - & - & + & + & - & + & 0 & - \\ + & - & - & + & - & + & 0 & - & + & + & + & - \\ - & + & - & 0 & - & - & + & - & + & - & - & - \\ + & + & - & - & + & 0 & - & - & - & + & - & - \\ + & 0 & - & + & + & - & + & - & - & - & + & + \\ 0 & + & + & + & + & + & + & - & + & + & - & + \end{bmatrix}$$

$$\begin{bmatrix} + & + & 0 & - & + & + & + & + & - & + & - & + \\ + & - & - & - & - & + & - & - & + & + & 0 & + \\ + & - & + & + & + & - & - & - & - & 0 & - & + \\ + & + & - & + & - & - & - & + & 0 & + & - & - \\ + & + & + & - & - & - & + & - & + & - & - & 0 \\ + & 0 & + & + & - & + & + & - & - & + & + & - \\ + & - & - & 0 & - & - & + & + & - & - & + & + \\ 0 & - & + & - & - & + & - & + & - & - & - & - \\ - & - & + & + & - & 0 & + & + & + & + & - & + \\ - & - & - & - & 0 & - & + & - & - & + & - & - \\ + & - & - & + & + & + & + & 0 & + & - & - & - \\ - & + & - & + & - & + & 0 & - & - & - & - & + \end{bmatrix}$$

We can see E_1, E_2 *and* E_3 in the upper left 7×7 submatrix of the matrices above, respectively. □

Remark 1 *The above proof specifies that every submatrix of* E_1, E_2 *and* E_3 *exists in* $W(12,11)$*, too. Let us denote by* A_1 *the upper left* 4×4 *block of* E_1 *and by* A_2 *the upper left* 4×4 *block of* E_2 *(or* E_3*). So, we have that* A_1 *and* A_2 *always exist in a* $W(12,11)$*, which means actually that the* 4×4 *minor of the* $W(12,11)$ *equals to* $detA_1 = 16$ *or* $detA_2 = 12$.

Next, we tried to extend the 4×4 matrices A_1 and A_2 to all possible 5×5 matrices. It is interesting to specify all possible 5×5 matrices M with elements $(0, +, -)$ that contain the matrices A_1 or A_2 and also have the maximum possible values of the determinant which for the 5×5 matrices are given in Lemma 1. The matrices with determinants that don't appear in the next tables couldn't be extended to a $W(12,11)$. We managed to show this with application of the Algorithm Extend [**?**]. We found the following results:

Extension of matrix A_1

det	18	20	22	24	26	28	30	32	36	40	48
matrices	0	30	0	42	0	42	0	81	21	18	3

Table 1

Extension of matrix A_2

det	14	16	18	20	22	24	26	28	30	32	36	40	48
matrices	48	108	48	0	10	61	4	18	10	12	11	3	0

Table 2

Tables 1 and 2 show the number of matrices which occurred as extensions of A_1 and A_2 with the corresponding determinant values. For odd values of determinants there weren't any matrices found. Tables 3, 4, 5 and 6 display the determinant value, the matrix from which the extension comes, the total number of possible extensions with the required determinant found, the numbers and the names of the H-equivalent representative matrices.

det	matrix	found	equiv	matrices
48	A_1	3	1	C_1
40	A_1	18	1	C_2
36	A_1	21	1	C_3
32	A_1	81	4	$C_4,\ C_5,\ C_6,\ C_7$
40	A_2	3	1	C_8
36	A_2	11	2	$C_9,\ C_{10}$

Table 3

We have checked with Algorithm Extend that from the matrices above only $C_1,\ C_8$ *and* C_9 with determinants 48, 40 *and* 36 respectively can be extended to a $W(12,11)$. So, we try to extend them to all possible 6×6 matrices with elements $(0,+,-)$ that contain the matrices $C_1,\ C_8$ *or* C_9 in the upper left 5×5 corner and also have the maximum possible values of the determinant which for the 6×6 matrices are given in Lemma 1. The values of determinants, for which have been found extensions, are presented in the following table:

det	matrix	found	equiv	matrices
160	C_1	4	1	D_1
144	C_1	21	3	
	C_8	3		
	C_9	1	1	D_2
136	C_1	1		
	C_8	4	1	D_3
120	C_1	45		
	C_8	11		
	C_9	2	1	D_4

Table 4

From the matrices above $D_1,\ D_2,\ D_3$ *and* D_4 with determinants 160, 144, 136 *and* 120 respectively can be extended to a $W(12,11)$. So, we try to extend them to all possible 7×7 matrices with elements $(0,+,-)$ that contain the matrices $D_1,\ D_2,\ D_3$ *and* D_4 in the upper left 6×6 corner and also have the maximum possible values of the determinant which for the 7×7 matrices are given in Lemma 1. The values of determinants, for which have been found extensions, are presented in the following table:

det	matrix	found	equiv	matrices
528	D_1	6	1	E_1
	D_2	4	1	E_2
	D_3	2	2	
440	D_3	7	7	$E_{3,\ldots,9}$
396	D_2	46	17	
	D_3	4	4	
	D_4	2	1	E_{10}

Table 5

From the matrices above only $E_1,\ E_2,$ *and* E_3 can be extended to a $W(12,11)$. With the same logic we proceed with the 8×8 extensions.

det	matrix	found	equiv	matrices
1936	E_1	4	2	$F_1,\ F_2$
	E_2	12	1	F_3
	E_3	1	1	F_4
1452	E_3	3	3	$F_{4,\ldots,6}$

Table 6

From the matrices above only F_1 *and* F_4 can be extended to a $W(12,11)$. This result was tested with Algorithm Extend.

A very helpful tool, which casts off many matrices by determining that they cannot be extended to a $W(n,n-1)$, is the following proposition of Goethals-Seidel:

Test for completion of a $W(n,n-1)$

Proposition 1 *Let A be a $W(n,n-1)$. Then A is H-equivalent with a matrix B which has zero diagonal entries and satisfies*

$$BB^T = (-1)^{\frac{n+2}{2}} I$$

Next we will examine if a matrix of order k with entries $0, \pm 1$ can be extended to a $W(n,n-1)$, by carrying out the following steps:

1. Exchange rows and columns so that the 0's are on the diagonal;
2. Multiply column with -1 so that all the entries of the first line are $+1$;
3. Multiply rows with -1 so that all the entries of the first column are $(-1)^{\frac{n+2}{2}}$;
4. Check if the resulting square matrix C, which contains all the 0's, satisfies

$$CC^T = (-1)^{\frac{n+2}{2}} I$$

If the matrix C doesn't satisfy this relationship, it can't be completed to a $W(n,n-1)$. If the matrix C satisfies the test, then it is possible that it can be completed to a $W(n,n-1)$.

In [**?**] was given the Algorithm Extend for extending a $k \times k$ $(0,+,-)$ matrix to $W(n,n-1)$. The algorithm specifies for a $k \times k$ matrix A its extension, if it exists, to a $W(n,n-1)$. Throughout this paper, wherever it is mentioned that a matrix can or can't be extended to a $W(12,11)$, it is meant by making use of this algorithm. Here we illustrate an example of the application of the algorithm for a $W(12,11)$.

Implementation of the Algorithm Extend

We apply the algorithm for k=5, n=12 .
Steps of the algorithm

1. We start with

$$A = \begin{bmatrix} + & + & + & + & + \\ + & - & + & - & - \\ + & - & - & + & + \\ + & + & - & - & + \\ + & + & - & + & - \end{bmatrix};$$

2. The first row and column is completed, without loss of generality, so that the property of a $W(12,11)$ having exactly one zero in each row and column is preserved. The software package fills with zeros the rest of the entries of the required 12×12 matrix;

$$A = \begin{bmatrix} + & + & + & + & + & + & + & 0 & + & + & - & - \\ + & - & + & - & - & & & & & & & \\ + & - & - & + & + & & & & & & & \\ + & + & - & - & + & & & & & & & \\ + & + & - & + & - & & & & & & & \\ + & & & & & & & & & & & \\ + & & & & & & & & & & & \\ 0 & & & & & & & & & & & \\ - & & & & & & & & & & & \\ - & & & & & & & & & & & \\ + & & & & & & & & & & & \\ - & & & & & & & & & & & \end{bmatrix}.$$

3. As before, the algorithm completes the second row and the second column in two ways, because the elements a and b beside the 0 of the first row and column respectively below can take both values ± 1;

$$A = \begin{bmatrix} + & + & + & + & + & + & + & 0 & + & + & - & - \\ + & - & + & - & - & - & - & - & a & + & - & 0 \\ + & - & - & + & + & & & & & & & \\ + & + & - & - & + & & & & & & & \\ + & + & - & + & - & & & & & & & \\ + & + & & & & & & & & & & \\ + & - & & & & & & & & & & \\ 0 & b & & & & & & & & & & \\ - & + & & & & & & & & & & \\ - & - & & & & & & & & & & \\ + & - & & & & & & & & & & \\ - & 0 & & & & & & & & & & \end{bmatrix}.$$

4. The algorithm takes as input this matrix A and finds all possible completions for rows 3-5 (columns 6-12), so that every row has exactly one zero, every column has at most

one zero and the inner product of every two distinct rows is zero. If many ways have been found to complete rows 3-5, the algorithm keeps as a result the first solution found;

$$A = \begin{bmatrix} + & + & + & + & + & + & + & 0 & + & + & - & - \\ + & - & + & - & - & - & - & - & - & + & - & 0 \\ + & - & - & + & + & + & - & - & 0 & - & - & + \\ + & + & - & - & + & - & 0 & - & + & - & + & - \\ + & + & - & + & - & - & - & + & - & 0 & - & - \\ + & + & & & & & & & & & & \\ + & - & & & & & & & & & & \\ 0 & + & & & & & & & & & & \\ - & + & & & & & & & & & & \\ - & - & & & & & & & & & & \\ + & - & & & & & & & & & & \\ - & 0 & & & & & & & & & & \end{bmatrix}.$$

5. The algorithm finds all possible completions for columns 3-5 (rows 6-12) in the same way it has done with the rows 3-5;

$$A = \begin{bmatrix} + & + & + & + & + & + & + & 0 & + & + & - & - \\ + & - & + & - & - & - & - & - & - & + & - & 0 \\ + & - & - & + & + & + & - & - & 0 & - & - & + \\ + & + & - & - & + & - & 0 & - & + & - & + & - \\ + & + & - & + & - & - & - & + & - & 0 & - & - \\ + & + & - & - & - & & & & & & & \\ + & - & - & + & 0 & & & & & & & \\ 0 & + & + & + & + & & & & & & & \\ - & + & 0 & + & + & & & & & & & \\ - & - & - & 0 & + & & & & & & & \\ + & - & + & + & - & & & & & & & \\ - & 0 & - & + & - & & & & & & & \end{bmatrix}.$$

6. The algorithm tries to complete,if possible, the rows 6-12(columns 6-12) in the same way as before;

$$A = \begin{bmatrix} + & + & + & + & + & + & + & 0 & + & + & - & - \\ + & - & + & - & - & - & - & - & + & + & - & 0 \\ + & - & - & + & + & + & - & - & 0 & - & - & + \\ + & + & - & - & + & - & 0 & - & + & - & + & - \\ + & + & - & + & - & - & - & + & - & 0 & - & - \\ + & + & - & - & - & + & + & - & - & + & 0 & + \\ + & - & - & + & 0 & - & + & + & + & + & + & + \\ 0 & + & + & + & + & - & - & - & - & + & + & + \\ - & + & 0 & + & - & - & + & - & + & - & - & + \\ - & - & - & 0 & + & - & + & - & - & + & - & - \\ + & - & + & + & - & 0 & + & - & - & - & + & - \\ - & 0 & - & + & - & + & - & - & + & + & + & - \end{bmatrix}.$$

7. Finally, if matrix A could be extended, the algorithm gives the completed matrix $W(12,11)$ and verifies whether the relationship $AA^T = 11I_{12}$ is valid. □

Using the above algorithm we can prove the following propositions:

Proposition 2 *$W(5) = 48$ or 40 or 36 for a $W(12,11)$*

Proof. We must show that from all the matrices in Tables 1 and 2, only the ones with determinant 48 *or* 40 *or* 36 can be extended to a $W(12,11)$. By using Algorithm Extend for $k = 5$, $n = 12$ and by testing all 5×5 matrices that have been found in Tables 1 and 2, we found that only the following matrices with determinants 48 *or* 40 *or* 36 can be extended to a $W(12,11)$.

$$C_1 = \begin{bmatrix} + & + & + & + & + \\ + & - & + & - & - \\ + & - & - & + & + \\ + & + & - & - & + \\ + & + & - & + & - \end{bmatrix}$$

$$C_8 = \begin{bmatrix} + & + & 0 & - & + \\ + & - & - & - & - \\ + & - & + & + & + \\ + & + & - & + & - \\ + & + & + & - & - \end{bmatrix}$$

$$C_9 = \begin{bmatrix} + & + & 0 & - & + \\ + & - & - & - & 0 \\ + & - & + & + & + \\ + & + & - & + & - \\ + & + & + & - & - \end{bmatrix}$$

By taking into consideration Lemma **??** and Remark 1, we have that the matrices C_1, C_8 *and* C_9 always exist in a $W(12,11)$ because C_1, C_8 and C_9 are the upper left 5×5 blocks of E_1, E_3 and E_2 respectively. Hence, since we have that C_1, C_8 *and* C_9 always exist in a $W(12,11)$, we can conclude that $W(5) = 48$ *or* 40 *or* 36 for a $W(12,11)$. □

Similarly, we can conclude the following results:

1. $W(6) = 160$ *or* 144 *or* 136 *or*120 for a $W(12,11)$
2. $W(7) = 528$ *or* 440 for a $W(12,11)$
3. $W(8) = 1936$ *or* 1452 for a $W(12,11)$

3 Specification of pivot patterns

For the conference matrix $W(n,n-1)$ since $WW^T = (n-1)I$ we have that $det(W) = (n-1)^{\frac{n}{2}}$. In [?], [?] have been proved the following three results:

Proposition 3 *Let W be a CP skew and symmetric or conference matrix of order n. Then the $(n-1)\times(n-1)$ minors are: $W(n-1)=(n-1)^{\frac{n}{2}-1}$.*

Proposition 4 *Let W be a CP skew and symmetric conference matrix of order n. Then the $(n-2)\times(n-2)$ minors are $W(n-2)=2(n-1)^{\frac{n}{2}-2}$.*

Theorem 2 *When Gaussian Elimination is applied on a CP skew and symmetric conference matrix W of order n the last three pivots are $n-1$, $\frac{n-1}{2}$ and $\frac{n-1}{2}$ or $n-1$.*

Notation. In [?] was proposed an algorithm for finding values of minors of weighing matrices. This algorithm will be useful in our theory for the calculation of the 9×9 minor. Any $W=W(n,n-1)$ matrix can be written:

$$W=\left[\begin{array}{cc} M & U_j \\ \varepsilon U_j^T & C \end{array}\right],$$

where M,C are $j\times j$ and $(n-j)\times(n-j)$ matrices respectively, with diagonal entries all 0, such that $M=\varepsilon M^T$ and $C=\varepsilon C^T$, $\varepsilon=(-1)^{\frac{n-2}{2}}$. U_j is the matrix of order $j\times(n-j)$, which has 1's in its first row and the rest of its elements are ± 1. Then, according to the algorithm described in [?], we have a formula for calculating $detC$, which is in fact the value of the $(n-j)\times(n-j)$ minor.

Now we are interested in calculating the 9×9 minor of $W(12,11)$. After applying the algorithm for $n=12$ and $j=3$ we get:

$$W(9)=5324\ for\ a\ W(12,11)$$

Remark 2 *We wanted to make use of a more sophisticated technique in order to calculate the 9×9 minor because the one with the extensions requires a lot of time due to the complexity of the algorithms used on the PC. However, we wanted to demonstrate this technique until the 8×8 minor because it reveals more results, as we can see in Tables 1-6. The algorithm for finding values of minors of weighing matrices, although it provides less results, is more efficient in practice and can be used more easily for larger dimensions n.*

Lemma 6 *The pivot patterns of the $W(12,11)$ are*

$$(1,2,2,3,\tfrac{10}{3},\tfrac{17}{5},\tfrac{11}{17/5},\tfrac{11}{5/2},\tfrac{11}{4},\tfrac{11}{2},\tfrac{11}{2},11)\ \text{or}$$
$$(1,2,2,4,3,\tfrac{10}{3},\tfrac{11}{10/3},\tfrac{11}{3},\tfrac{11}{4},\tfrac{11}{2},\tfrac{11}{2},11)\ \text{or}$$
$$(1,2,2,3,3,4,\tfrac{11}{3},\tfrac{11}{3},\tfrac{11}{4},\tfrac{11}{2},\tfrac{11}{2},11)$$

Proof. We have shown that for every $W(n,n-1)$ the first four pivots are $1,2,2,3$ or 4. From Theorem **??** we also have that

$$p_{12}=11,\quad p_{11}=\frac{11}{2}.$$

We have

$$W(5)=48\ or\ 40\ or\ 36\ \text{for a}\ W(12,11)$$

The 5×5 matrix with determinant 48 contains in the upper left corner the 4×4 matrix A_1 with determinant 16. The 5×5 matrices with determinant 40 and 36 contain in the upper left corner the 4×4 matrix A_2 with determinant 12. So, the fifth pivot of $W(12,11)$ can be calculated using relationship (1):

$$p_5 = \frac{w(5)}{w(4)} \Rightarrow p_5 = \frac{48}{16} \text{ or } \frac{36}{12} \text{ or } \frac{40}{12} \Rightarrow p_5 = 3 \text{ or } \frac{10}{3}.$$

With the same logic, we go on to the sixth pivot: we have

$$W(6) = 160 \textit{ or } 144 \textit{ or } 136 \textit{ or} 120 \text{ for a } W(12,11)$$

The 6×6 matrices with determinant 160 contain in the upper left corner the 5×5 matrix with determinant 48. The 6×6 matrices with determinant 144 contain in the upper left corner the 5×5 matrix with determinant 36. The 6×6 matrices with determinant 136 contain in the upper left corner the 5×5 matrix with determinant 40. The 6×6 matrices with determinant 120 contain in the upper left corner the 5×5 matrix with determinant 36. So, the sixth pivot of $W(12,11)$ can be calculated using relationship (1):

$$p_6 = \frac{w(6)}{w(5)} \Rightarrow p_6 = \frac{160}{48} \text{ or } \frac{144}{36} \text{ or } \frac{136}{40} \text{ or } \frac{120}{36} \Rightarrow p_6 = 4 \text{ or } \frac{10}{3} \text{ or } \frac{17}{5}.$$

About the seventh pivot: we have

$$W(7) = 528 \textit{ or } 440 \text{ for a } W(12,11)$$

The 7×7 matrices with determinant 528 contain in the upper left corner the 6×6 matrices with determinant 160 and 144. The 7×7 matrices with determinant 440 contain in the upper left corner the 6×6 matrix with determinant 136. So, the seventh pivot of $W(12,11)$ can be calculated using relationship (1):

$$p_7 = \frac{w(7)}{w(6)} \Rightarrow p_7 = \frac{528}{160} \text{ or } \frac{528}{144} \text{ or } \frac{440}{136} \Rightarrow p_7 = \frac{11}{17/5} \text{ or } \frac{11}{10/3} \text{ or } \frac{11}{3}.$$

About the eighth pivot: we have

$$W(8) = 1936 \textit{ or } 1452 \text{ for a } W(12,11)$$

The 8×8 matrices with determinant 1936 contain in the upper left corner the 6×6 matrices with determinants 528 and 440. So, the eighth pivot of $W(12,11)$ can be calculated using relationship (1):

$$p_8 = \frac{w(8)}{w(7)} \Rightarrow p_8 = \frac{1936}{528} \text{ or } \frac{1936}{440} \Rightarrow p_7 = \frac{11}{5/2} \text{ or } \frac{11}{3}.$$

Since we have W(9)=5324 for a W(12,11), we can calculate p_9:

$$p_9 = \frac{w(9)}{w(8)} \Rightarrow p_9 = \frac{5324}{1936} \Rightarrow p_9 = \frac{11}{4}.$$

$$p_{11} = \frac{det(W(12,11))}{\prod_{i=1_{i\neq 10}}^{12} p_i} = \frac{11^6}{1\cdot 2\cdot 2\cdot 3\cdot \frac{10}{3}\cdot \frac{17}{5}\cdot \frac{11}{17/5}\cdot \frac{11}{5/2}\cdot \frac{11}{4}\cdot \frac{11}{2}\cdot 11} \quad or \quad \frac{11^6}{1\cdot 2\cdot 2\cdot 4\cdot 3\cdot \frac{10}{3}\cdot \frac{11}{10/3}\cdot \frac{11}{3}\cdot \frac{11}{4}\cdot \frac{11}{2}\cdot 11} \quad or$$

$$\frac{11^6}{1\cdot 2\cdot 2\cdot 3\cdot 3\cdot 4\cdot \frac{11}{3}\cdot \frac{11}{3}\cdot \frac{11}{4}\cdot \frac{11}{2}\cdot 11} \Rightarrow p_{10} = \frac{11}{2}$$

□

Theorem 3 *The growth factor of $W(12,11)$ is 11.*

Proof. The result follows obviously from Lemma **??** and from the definitions for the growth factor and the pivots given in the introduction. □

Remark 3 *The $W(12,11)$ matrices in the proof of Lemma **??** have pivot patterns* $(1,2,2,3,3,4,\frac{11}{3},\frac{11}{3},\frac{11}{4},\frac{11}{2},\frac{11}{2},11)$, $(1,2,2,4,3,\frac{10}{3},\frac{11}{10/3},\frac{11}{3},\frac{11}{4},\frac{11}{2},\frac{11}{2},11)$ *and* $(1,2,2,3,\frac{10}{3},\frac{17}{5},\frac{11}{17/5},\frac{11}{5/2},\frac{11}{4},\frac{11}{2},\frac{11}{2},11)$ *respectively.*

□

References

[1] A.M. Cohen, A note on pivot size in Gaussian elimination, *Linear Algebra Appl.*, 8 (1974), 361-368.

[2] C.W. Cryer, Pivot size in Gaussian elimination, *Numer. Math.*, 12 (1968), 335-345.

[3] J. Day, and B. Peterson, Growth in Gaussian elimination, *Amer. Math. Monthly*, 95 (1988), 489-513.

[4] A. Edelman, and W. Mascarenhas, On the complete pivoting conjecture for a Hadamard matrix of order 12, *Linear and Multilinear Algebra*, 38 (1995), 181-187.

[5] A.V.Geramita, and J.Seberry, *Orthogonal Designs: Quadratic Forms and Hadamard Matrices*, Marcel Dekker, New York-Basel, 1979.

[6] N. Gould, On growth in Gaussian elimination with pivoting, *SIAM J. Matrix Anal. Appl.*, 12 (1991), 354-361.

[7] N. J. Higham, Accuracy and Stability of Numerical Algorithms, *SIAM*, 2002.

[8] C. Kravvaritis, M. Mitrouli and J. Seberry, On the growth problem for skew and symmetric conference matrices, accepted for publication in *Linear Algebra and its Appl.*.

[9] C. Kravvaritis, E. Lappas and M. Mitrouli, An algorithm to find values of minors of weighing matrices, Lecture Notes in Computer Science, 3401, 375-382, *Springer Verlag.*

[10] C. Koukouvinos, M. Mitrouli and J. Seberry, Growth in Gaussian elimination for weighing matrices, $W(n,n-1)$, *Linear Algebra and its Appl.*, 306 (2000), 189-202.

[11] L. N. Trefethen and D. Bau,III, Numerical Linear Algebra, SIAM, 1997.

[12] L. N. Trefethen and R. S. Schreiber, Average-case stability of Gaussian elimination, *SIAM J. Matrix Anal. Appl.* 11 (1990), 335-360.

[13] J. H. Wilkinson, Error analysis of direct methods of matrix inversion, *J. Assoc. Comput. Mach.*, 8 (1961), 281-330.

[14] J. H. Wilkinson, Rounding Errors in Algebraic Processes, *Her Majesty's Stationery Office*, London, 1963.

[15] J. H. Wilkinson, The Algebraic Eigenvalue Problem, *Oxford University Press*, London, 1988.