Orthogonal Designs with Quaternion Elements

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Abstract

We introduce orthogonal designs with quaternion elements and show their existence. In future work we explore these applications to signal processing.

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1 Introduction

The introduction of Space-Time Codes to harness the benefits of combined space and time diversity was a major step in moving the capacity of wireless communication systems towards the theoretical limits. The technique has been adopted in the 3G standard in the form of an Alamouti code [1] and in the newly proposed standard for wireless LANs IEEE 802.11n [2]. Application of other forms of diversity together with STCs can improve this even further. The two obvious techniques to be considered together with STCs are frequency diversity and polarisation diversity.

Polarisation diversity has been widely studied in the past, e.g. [3] with an assessment of the diversity gain under Rayleigh fading presented in [4]. This form of diversity is usually considered separately from the others and there is no well-known mechanism of utilising it jointly with the other forms rather than through a simple concatenation. In [5], Isaeva and Sarytchev showed that polarisation state can be nicely modeled by means of quaternion representation. Hence, an orthogonal design with the quaternion elements can become a basis of an Orthogonal Space-Time-Polarization code where polarisation diversity can be considered jointly with space and time diversities.

An *Hadamard matrix* H of order n is a square $(1,-1)$ matrix having inner product of distinct rows zero. Hence $HH^T = nI_n$. We note that $n = 1, 2$ or $n \equiv 0 \pmod 4$.

Traditionally an *orthogonal design* of order n and type $(s_1, s_2, \ldots, s_u)$ $(s_i > 0)$, denoted $OD(n; s_1, s_2, \ldots, s_u)$, on the commuting variables $x_1, x_2, \ldots, x_u$ is an $n \times n$ matrix A with entries from $\{0, \pm x_1, \pm x_2, \ldots, \pm x_u\}$ such that

$$AA^T = (\sum_{i=1}^{u} s_i x_i^2) I_n.$$

Alternatively, the rows of A are formally orthogonal and each row has precisely s_i entries of the type $\pm x_i$. In [6], where this was first defined, it was mentioned that

$$A^T A = (\sum_{i=1}^{u} s_i x_i^2) I_n$$

and so our alternative description of A applies equally well to the columns of A. It was also shown in [6] that $u \leq \rho(n)$, where $\rho(n)$ (Radon's function) is defined by $\rho(n) = 8c + 2^d$, when $n = 2^a b$, b odd, $a = 4c + d$, $0 \leq d < 4$.

Orthogonal designs with complex elements are discussed in [7].

We now consider the quaternion elements $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$, where

$$\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = -1, \text{ and } \mathbf{ij} = \mathbf{k}, \mathbf{jk} = \mathbf{i}, \mathbf{ki} = \mathbf{j}, \text{ with 1 the unit.}$$

We will say a number $\mathbf{a}$ is a *quaternion number* if

$$\begin{aligned} \mathbf{a} &= a_1 + a_2\mathbf{i} + a_3\mathbf{j} + a_4\mathbf{k} \\ &= (a_1 + a_2\mathbf{i}) + (a_3 + a_4\mathbf{i})\mathbf{j}, \end{aligned}$$

where a_i, $i = 1, \ldots, 4$ are real numbers. We say a variable $\mathbf{a}$ is a *quaternion variable* if $\mathbf{a} = a_1 + a_2\mathbf{i} + a_3\mathbf{j} + a_4\mathbf{k}$, where a_i, $i = 1, \ldots, 4$ are real variables.

We define the *quaternion transform* q^Q of a quaternion q by analogy with complex conjugation and hermitian transforms. q^Q is the quaternion such that $q^Q q = q q^Q = 1$. For example, $\mathbf{i}^Q = -\mathbf{i}$. When q is real, $q^Q = q$.

Let q, r be quaternions. We define the quaternion transform of their product as follows: $(qr)^Q = r^Q q^Q$.

Let $\mathbf{a}$ be a quaternion number (or variable). Then its quaternion transform $\mathbf{a}^Q$ is

$$\begin{aligned}
\mathbf{a}^Q &= (a_1 + a_2\mathbf{i} + a_3\mathbf{j} + a_4\mathbf{k})^Q \\
&= a_1{}^Q + (a_2\mathbf{i})^Q + (a_3\mathbf{j})^Q + (a_4\mathbf{k})^Q \\
&= a_1 + \mathbf{i}^Q a_2 + \mathbf{j}^Q a_3 + \mathbf{k}^Q a_4 \\
&= a_1 - \mathbf{i}a_2 - \mathbf{j}a_3 - \mathbf{k}a_4 \\
&= a_1 - a_2\mathbf{i} - a_3\mathbf{j} - a_4\mathbf{k}
\end{aligned}$$

Further we define the inner product of two quaternion variables $\mathbf{a} = a_1 + a_2\mathbf{i} + a_3\mathbf{j} + a_4\mathbf{k}$, and $\mathbf{b} = b_1 + b_2\mathbf{i} + b_3\mathbf{j} + b_4\mathbf{k}$, as

$$\begin{aligned}
\mathbf{a} \cdot \mathbf{b} &= \mathbf{a}\mathbf{b}^Q \\
&= (a_1 + a_2\mathbf{i} + a_3\mathbf{j} + a_4\mathbf{k})(b_1 - b_2\mathbf{i} - b_3\mathbf{j} - b_4\mathbf{k}) \\
&= (a_1b_1 + a_2b_2 + a_3b_3 + a_4b_4) \\
&\quad + (-a_1b_2 + a_2b_1 - a_3b_4 + a_4b_3)\mathbf{i} \\
&\quad + (-a_1b_3 + a_2b_4 + a_3b_1 - a_4b_2)\mathbf{j} \\
&\quad + (-a_1b_4 - a_2b_3 + a_3b_2 + a_4b_1)\mathbf{k}.
\end{aligned}$$

We define the quaternion transform of a matrix $A = [a_{ij}]$ as $A^Q = \left[a_{ji}{}^Q\right]$.

2 Preliminary results

Lemma 1 *Let* $\mathbf{a}$ *be a quaternion variable (or number) then* $\mathbf{a}\mathbf{a}^Q = \sum_{i=1}^{4} a_i^2$, *which is real.*

Lemma 2 *Let* $\mathbf{a}$ *be a quaternion variable (or number). Then* $\mathbf{a} + \mathbf{a}^Q$ *is real.*

Lemma 3 *Let* $\mathbf{a}$ *and* $\mathbf{b}$ *be quaternion variables (or numbers) then* $\mathbf{ab}^Q = \mathbf{ba}^Q$ *only if*

$$\begin{aligned} -a_1b_2 + a_2b_1 - a_3b_4 + a_4b_3 &= \\ -a_1b_3 + a_2b_4 + a_3b_1 - a_4b_2 &= \\ -a_1b_4 - a_2b_3 + a_3b_2 + a_4b_1 &= 0. \end{aligned}$$

Proof. We expand $\mathbf{ab}^Q$ and $\mathbf{ba}^Q$ and equate the terms in $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ to get the result. □

We now define a *quaternion orthogonal design* of order n and type $(s_1, s_2, \ldots, s_u)$ $(s_i > 0)$, denoted $QOD(n; s_1, s_2, \ldots, s_u)$, on the quaternion commuting variables $x_1, x_2, \ldots, x_u$ as an $n \times n$ matrix A with entries from $\{0, q_1x_1, q_2x_2, \ldots, q_ux_u\}$, where each q_i is a linear combination of $\{\pm 1, \ \pm \mathbf{i}, \ \pm \mathbf{j}, \ \pm \mathbf{k}\}$ such that

$$AA^Q = (\sum_{i=1}^{u} s_i x_i^2) I_n.$$

Example 1 Suppose $\mathbf{a}$ and $\mathbf{b}$ are quaternion variables such that

$$a_1b_3 - a_2b_4 - a_3b_1 + a_4b_2 = 0.$$

Then $D = \begin{bmatrix} \mathbf{a} & \mathbf{jb} \\ \mathbf{ib} & -\mathbf{ka} \end{bmatrix}$ is a $QOD(2; 1, 1)$. This follows as

$$\begin{aligned} DD^Q &= \begin{bmatrix} \mathbf{a} & \mathbf{jb} \\ \mathbf{ib} & -\mathbf{ka} \end{bmatrix} \begin{bmatrix} \mathbf{a}^Q & -\mathbf{b}^Q\mathbf{i} \\ -\mathbf{b}^Q\mathbf{j} & \mathbf{a}^Q\mathbf{k} \end{bmatrix} \\ &= \begin{bmatrix} \mathbf{aa}^Q + \mathbf{bb}^Q & -\mathbf{iab}^Q + \mathbf{iba}^Q \\ \mathbf{iba}^Q - \mathbf{iab}^Q & \mathbf{aa}^Q + \mathbf{bb}^Q \end{bmatrix} \\ &= (\mathbf{aa}^Q + \mathbf{bb}^Q) I_2. \end{aligned}$$

□

Example 2 Suppose $\mathbf{a}, \mathbf{b}, \mathbf{x}, \mathbf{y}$ are quaternion variables such that

$$\begin{aligned} \mathbf{ax}^Q &= \mathbf{xa}^Q, \\ \mathbf{by}^Q &= \mathbf{yb}^Q, \\ a_1y_2 - a_2y_1 - a_3y_4 + a_4y_3 &= 0, \text{ and} \qquad (1) \\ b_1x_2 - b_2x_1 - b_3x_4 + b_4x_3 &= 0. \qquad (2) \end{aligned}$$

Then the matrices

$$A = \begin{bmatrix} \mathbf{a} & \mathbf{bj} \\ \mathbf{bi} & -\mathbf{ak} \end{bmatrix}, B = \begin{bmatrix} \mathbf{x} & -\mathbf{yj} \\ \mathbf{yi} & \mathbf{xk} \end{bmatrix}$$

have the property that $AB^Q = BA^Q$. Such matrices, by analogy with the real case, will be called quaternion amicable matrices. Thus the matrices A, B are quaternion amicable orthogonal designs QAOD$(2; 1, 1; 1, 1)$.

Proof. Let $\mathbf{aiy}^Q = \alpha_1 + \alpha_2\mathbf{i} + \alpha_3\mathbf{j} + \alpha_4\mathbf{k}$. Then $(\mathbf{aiy}^Q)^Q = \alpha_1 - \alpha_2\mathbf{i} - \alpha_3\mathbf{j} - \alpha_4\mathbf{k}$. Now, $\mathbf{yia}^Q = -(\mathbf{aiy}^Q)^Q$. Hence $\mathbf{yia}^Q = -\alpha_1 + \alpha_2\mathbf{i} + \alpha_3\mathbf{j} + \alpha_4\mathbf{k}$. But $\alpha_1 = a_1y_2 - a_2y_1 - a_3y_4 + a_4y_3$. By equation (1), $\alpha_1 = 0$. So $\mathbf{aiy}^Q = \mathbf{yia}^Q$.

Likewise, it can be shown that $\mathbf{bix}^Q = \mathbf{xib}^Q$ by equation (2).

$$\begin{aligned} AB^Q &= \begin{bmatrix} \mathbf{a} & \mathbf{bj} \\ \mathbf{bi} & -\mathbf{ak} \end{bmatrix} \begin{bmatrix} \mathbf{x}^Q & -\mathbf{iy}^Q \\ \mathbf{jy}^Q & -\mathbf{kx}^Q \end{bmatrix} \\ &= \begin{bmatrix} \mathbf{ax}^Q - \mathbf{by}^Q & -\mathbf{aiy}^Q - \mathbf{bix}^Q \\ \mathbf{bix}^Q + \mathbf{aiy}^Q & \mathbf{by}^Q - \mathbf{ax}^Q \end{bmatrix} \\ &= \begin{bmatrix} \mathbf{xa}^Q - \mathbf{yb}^Q & -\mathbf{yia}^Q - \mathbf{xib}^Q \\ \mathbf{xib}^Q + \mathbf{yia}^Q & \mathbf{yb}^Q - \mathbf{xa}^Q \end{bmatrix} \\ &= \begin{bmatrix} \mathbf{x} & -\mathbf{yj} \\ \mathbf{yi} & \mathbf{xk} \end{bmatrix} \begin{bmatrix} \mathbf{a}^Q & -\mathbf{ib}^Q \\ -\mathbf{jb}^Q & \mathbf{ka}^Q \end{bmatrix} \\ &= BA^Q \end{aligned}$$

□

3 Conclusion

We have established the existence of quaternion orthogonal designs and quaternion amicable orthogonal designs. Their use in signal processing will be explained in future work.

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