

A new method for constructing T -matrices *

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Abstract

For every prime power $q \equiv 3 \pmod{8}$ we prove the existence of $(q; x, 0, y, y)$ -partitions of $GF(q)$ with $q = x^2 + 2y^2$ for some x, y , which are very useful for constructing SDS , T -matrices and Hadamard matrices. We discuss the transformations of $(q; x, 0, y, y)$ -partitions and, by using the partitions, construct generalized cyclotomic classes which have properties similar to those of classical cyclotomic classes. Thus we provide a new construction for T -matrices of order q^2 .

Keyword: T -matrices; Hadamard matrices; SDS ; $(q; x, 0, y, y)$ -partitions

1 Introduction

In 1965 L. Baumert and M. Hall, Jr [1] found a construction of Hadamard matrices of order $12n$ from known Williamson matrices of order n . Indeed, they gave the first example of T -matrices of order 3. Many attempts were made to generalize this array, but none were successful until in 1971 L. R. Welch found a Baumert-Hall array of order 5. In 1972, Joan Cooper and Jennifer Seberry Wallis [4] gave the first definition of T -matrices. R. J. Turyn [11] proposed the notion of 4δ codes and Turyn's sequences, using Golay sequences, he constructed an infinite class of T -matrices of order $2^i 10^j 26^k + 1$ for $i, j, k \geq 0$. Since Turyn's sequences are very restrictive and very few are known, Turyn, then J. Seberry, C. H. Yang, C. Koukouvinos, etc. investigated base sequences instead and found a large number of existent cases (for details see [10]). In 1984 M. Y. Xia [12] proposed the idea of C -partitions on an Abelian group, and then an infinite family of C -partitions on $GF(q^2)$ with q prime power $\equiv 3 \pmod{8}$ was found [14]. Now the construction of C -partitions in this paper is more general and yields many new T -matrices and Hadamard matrices.

Let G be an Abelian group of order v . We denote the group operation by multiplication. Subsets D_1, \dots, D_r of G are called r - $\{v; |D_1|, \dots, |D_r|; \lambda\}$ supplementary difference sets (SDS), if for every nonidentity element g in G , there are exactly λ elements (d, d') in $D_1 \times D_1$, or $D_2 \times D_2, \dots$, or $D_r \times D_r$ such that $gd' = d$.

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It is convenient to use the group ring $Z[G]$ of the group G over the ring Z of rational integers with the addition and multiplication. Here the elements of $Z[G]$ are of the form

$$a_1g_1 + a_2g_2 + \cdots + a_vg_v, a_i \in Z, g_i \in G.$$

In $Z[G]$ the addition $+$ is given by the rule

$$\left(\sum_g a(g)g\right) + \left(\sum_g b(g)g\right) = \sum_g (a(g) + b(g))g.$$

The multiplication in $Z[G]$ is given by the rule

$$\left(\sum_g a(g)g\right)\left(\sum_h b(h)h\right) = \sum_k \left(\sum_{gh=k} a(g)b(h)\right)k.$$

For any subset A of G , we define an element

$$\sum_{g \in A} g \in Z[G],$$

and by abusing the notation we will denote it by A .

Let $A, B \subset G$ and t be an integer. We define

$$B^{(t)} = \sum_{b \in B} b^t \in Z[G], \quad AB^{(-1)} = \sum_{a \in A, b \in B} ab^{-1} \in Z[G]$$

and denote

$$\Delta A = AA^{(-1)}, \Delta(A, B) = AB^{(-1)} + BA^{(-1)}.$$

If $A = \emptyset$, we define

$$\Delta \emptyset = 0, \Delta(\emptyset, B) = 0.$$

With this convention D_1, \dots, D_r being r - $\{v; |D_1|, \dots, |D_r|; \lambda\}$ *SDS* are equivalent to

$$\sum_{i=1}^r \Delta D_i = \left(\sum_{i=1}^r |D_i| - \lambda\right) + \lambda G.$$

If $r = 1$ the single *SDS* becomes a difference set (*DS*) in the usual sense. When $|D_1| = \dots = |D_r| = k$, we denote r - $\{v; |D_1|, \dots, |D_r|; \lambda\}$ by r - $\{v; k; \lambda\}$.

In the following we assume p is an odd prime, $r > 0$, and

$$q = p^r = 8m + 3 = x^2 + 2y^2 \tag{1}$$

with $x \equiv 1 \equiv y \pmod{2}$.

In this paper we propose the notion of $(q; x, 0, y, y)$ -partition of $GF(q)$ and prove its existence for some x, y satisfying (1). It provides a very useful method for constructing *SDS*, Hadamard matrices and *T*-matrices. Y. Q. Chen [3] constructed a partition of $GF(q^2)$. Then [15] generalized it from $GF(q^2)$ to $GF(q)$ with q prime power $\equiv 1 \pmod{4}$. Now we extend it to the case q prime power $\equiv 3 \pmod{8}$.

The rest of the paper is organized as follows. In section 2, we will partition the group $GF(q)$ into 8 subsets with certain desirable properties. In section 3, we use the partition obtained in Section 2 to define the generalized cyclotomic

classes and discuss their properties. In section 4, by using generalized cyclotomic classes, we will construct $4\text{-}\{q; (q-1)/2; q(q-2)\}SDS$, Hadamard matrices of order $4q^2$. In section 5 we show that there are lots of T -matrices of order q^2 .

Before we proceed further, we list the notations that will be used throughout this paper:

- q : a power of an odd prime p as in (1);
- $GF(q)$: the Galois field with q elements;
- $GF(q)^*$: the multiplicative group of $GF(q)$;
- S : the set of all nonzero squares of $GF(q)$;
- N : the set of all no squares of $GF(q)$;
- δ : a generator of $GF(q)^*$;
- $\text{Tr}q^n$: the absolute trace from $GF(q^n)$ to $GF(p)$;
- $\text{Tr}q^n/q$: the relative trace from $GF(q^n)$ to $GF(q)$;
- $\langle i, j \rangle$: the cyclotomy number.

Recall that the absolute trace $\text{Tr}q^n$ of an element $g \in GF(q^n)$ is defined as

$$\text{Tr}q^n(g) = \sum_{j=0}^{qn-1} g^{p^j} \in GF(p).$$

For the detailed discussion of absolute and relative trace maps of finite fields, we refer the reader to textbooks such as [6], [7] and [8]. The characters of the group $GF(q^n)$ are given by the following (see [9]). Let ξ be a fixed primitive p th root of unity, $\alpha, \beta \in GF(q^n)$, define a group homomorphism

$$\begin{aligned} \chi_\alpha &: GF(q^n) \rightarrow C^*, \\ \chi_\alpha(\beta) &= \xi^{\text{Tr}q^n(\alpha\beta)}, \end{aligned}$$

where C^* is the multiplicative group of nonzero complex numbers. These group homomorphisms can be easily extended to ring homomorphisms from $Z[GF(q^n)]$ to C . In order to show $A = B$ in $Z[GF(q^n)]$ by using the Fourier inversion formula, we need only to verify $\chi_\alpha(A) = \chi_\alpha(B)$ for every $\alpha \in GF(q^n)$.

2 $(q; x, 0, y, y)$ -Partitions

Let w be a solution of the irreducible polynomial $x^2 + 1$ over $GF(q)$. Then the set of all elements $\alpha w + \beta$, $\alpha, \beta \in GF(q)$, is $GF(q^2)$. It is well known that there is an element $g = \alpha w + \beta$, $\alpha, \beta \in GF(q)$, such that

$$GF(q^2)^* = \{g^k : k = 0, 1, \dots, q^2 - 2\}.$$

Let g be such an element and put

$$E_i = \{g^{8(2m+1)j+i} : j = 0, 1, \dots, 4m\}, i = 0, 1, \dots, 16m + 7.$$

It is easy to show that

$$E_0 = \{\delta^{2k} : k = 0, 1, \dots, 4m\} = S,$$

and

$$E_{8m+4} = \{\delta^{2k+1} : k = 0, 1, \dots, 4m\} = N.$$

For any $i, 1 \leq i < 16m + 8, i \neq 8m + 4$, write $g^i = \alpha w + \beta$, then $\alpha \neq 0$ and

$$\begin{aligned} E_i = g^i E_0 &= \left\{ \alpha \delta^{2k} w + \beta \delta^{2k} : j = 0, 1, \dots, 4m \right\} \\ &= \left\{ \alpha \delta^{2k} w + \alpha^{-1} \beta \alpha \delta^{2k} : j = 0, 1, \dots, 4m \right\} \\ &\triangleq \left\{ (\alpha \delta^{2k}, \alpha^{-1} \beta (\alpha \delta^{2k})) : j = 0, 1, \dots, 4m \right\}. \end{aligned}$$

So we can represent E_i by $\{(\eta, \gamma\eta) : \eta \in S\}$ or $\{(\eta, \gamma\eta) : \eta \in N\}$ according to $\alpha \in S$ or $\alpha \in N$. For convenience, we denote

$$E_0 = (0, S), \quad E_{8m+4} = (0, N)$$

and

$$\begin{aligned} \{(\eta, \gamma\eta) : \eta \in S\} &= (S, \gamma S), \\ \{(\eta, \gamma\eta) : \eta \in N\} &= (N, \gamma N). \end{aligned}$$

The partition given in the following theorem is the basis of the paper. It provides a useful method for constructing SDS , Hadamard matrices and T -matrices.

Theorem 1 *There exist eight subsets, X_1, \dots, X_8 , of $GF(q)$, q and m satisfy (1), such that*

$$|X_1| = |X_2| = m, \quad (2)$$

$$\{|X_3|, |X_4|\} = \left\{ m + \frac{1}{2}(1+y), m + \frac{1}{2}(1-y) \right\} = \{|X_7|, |X_8|\}, \quad (3)$$

$$\{|X_5|, |X_6|\} = \left\{ m + \frac{1}{2}(1+x), m + \frac{1}{2}(1-x) \right\}, \quad (4)$$

$$X_1 + \dots + X_8 = GF(q), \quad (5)$$

$$V = MU, \quad (6)$$

for some x, y satisfying (1), where

$$V = (X_1 N + X_2 S, X_1 S + X_2 N, \dots, X_7 N + X_8 S, X_7 S + X_8 N)', \quad (7)$$

$$U = (X_1, \dots, X_8)' \quad (8)$$

and

$$M = \begin{pmatrix} |X_1|-1 & |X_2|-1 & |X_3|-1 & |X_4|-1 & |X_5|-1 & |X_6|-1 & |X_7|-1 & |X_8|-1 \\ |X_2| & |X_1| & |X_4| & |X_3| & |X_6| & |X_5| & |X_8| & |X_7| \\ |X_4| & |X_3| & |X_1| & |X_2| & |X_7| & |X_8| & |X_6| & |X_5| \\ |X_3| & |X_4| & |X_2| & |X_1| & |X_8| & |X_7| & |X_5| & |X_6| \\ |X_6| & |X_5| & |X_8| & |X_7| & |X_1| & |X_2| & |X_3| & |X_4| \\ |X_5| & |X_6| & |X_7| & |X_8| & |X_2| & |X_1| & |X_4| & |X_3| \\ |X_8| & |X_7| & |X_5| & |X_6| & |X_4| & |X_3| & |X_1| & |X_2| \\ |X_7| & |X_8| & |X_6| & |X_5| & |X_3| & |X_4| & |X_2| & |X_1| \end{pmatrix}. \quad (9)$$

We call the partition satisfying (2)–(9) a $(q; x, 0, y, y)$ -partition.

Proof. Put

$$C_i = \left\{ g^k : k \equiv i \pmod{8} \right\}, \quad i = 0, 1, \dots, 7,$$

where g is a generator of $GF(q^2)$. It is clear that

$$C_i = \bigcup_{j=0}^{2m} E_{8j+i}, \quad i = 0, 1, \dots, 7.$$

Particularly, C_0 and $C_4 = g^{8m+4}C_0$ can be written in the forms

$$C_0 = (0, S) \cup \{(S, \gamma S), \gamma \in X_1\} \cup \{(N, \gamma N), \gamma \in X_2\}, \quad (10)$$

$$C_4 = (0, N) \cup \{(N, \gamma N), \gamma \in X_1\} \cup \{(S, \gamma S), \gamma \in X_2\} \quad (11)$$

for some subsets X_1 and X_2 of $GF(q)$. Obviously,

$$|X_1| + |X_2| = 2m. \quad (12)$$

For any $i, 1 \leq i \leq 2m$, write $g^{8i} = \alpha w + \beta (\in E_{8i})$ and $\alpha \neq 0$ for sure. Now

$$(g^{8i})^{8m+3} = g^{(16m+8)(4i-1)+8(2m+1-i)} \in E_{8(2m+1-i)}$$

and

$$(\alpha w + \beta)^{8m+3} = \alpha w^{8m+3} + \beta = -\alpha w + \beta,$$

so $\alpha(-\alpha) \in N$ and $\alpha^{-1}\beta + (-\alpha)^{-1}\beta = 0$. Therefore $\gamma = \alpha^{-1}\beta \in X_i$ if and only if $-\gamma \in X_{3-i}, i = 1, 2$. These facts, together with (12), show that

$$|X_1| = |X_2| = m \quad (13)$$

and

$$0 \notin X_1 \cup X_2. \quad (14)$$

Now take

$$X_5 = \{-\gamma^{-1} : \gamma \in (X_1 \cap N) \cup (X_2 \cap S)\}, \quad (15)$$

$$X_6 = \{0\} \cup \{-\gamma^{-1} : \gamma \in (X_1 \cap S) \cup (X_2 \cap N)\}. \quad (16)$$

Since $\{C_2, C_6\} = \{g^{4m+2}C_0, g^{12m+6}C_0\}$ and $\{E_{4m+2}, E_{12m+6}\} = \{(S, 0), (N, 0)\}$, so

$$\{C_2, C_6\} = \left\{ \bigcup_{\gamma \in X_5} (S, \gamma S) \bigcup \left(\bigcup_{\gamma \in X_6} (N, \gamma N) \right), \bigcup_{\gamma \in X_5} (N, \gamma N) \bigcup \left(\bigcup_{\gamma \in X_6} (S, \gamma S) \right) \right\}.$$

Without loss of generality, we can write

$$C_2 = \{(S, \gamma S), \gamma \in X_5\} \cup \{(N, \gamma N), \gamma \in X_6\}, \quad (17)$$

$$C_6 = \{(N, \gamma N), \gamma \in X_5\} \cup \{(S, \gamma S), \gamma \in X_6\}. \quad (18)$$

Clearly

$$|X_5| + |X_6| = 2m + 1. \quad (19)$$

Since

$$\{E_{2m+1}, E_{6m+3}, E_{10m+5}, E_{14m+7}\} = \{(S, -S), (N, N), (N, -N), (S, S)\},$$

it follows that

$$1, -1 \notin X_1 \cup X_2.$$

Define

$$X_3 = \{-1\} \cup \{-(\gamma-1)^{-1}(\gamma+1) : \gamma \in (X_1 \cap (S+1)) \cup (X_2 \cap (N+1))\},$$

$$X_4 = \{-(\gamma-1)^{-1}(\gamma+1) : \gamma \in (X_1 \cap (N+1)) \cup (X_2 \cap (S+1))\},$$

$$X_7 = \{(\gamma+1)^{-1}(\gamma-1) : \gamma \in (X_1 \cap (N-1)) \cup (X_2 \cap (S-1))\},$$

$$X_8 = \{1\} \cup \{(\gamma+1)^{-1}(\gamma-1) : \gamma \in (X_1 \cap (S-1)) \cup (X_2 \cap (N-1))\}.$$

Similarly to (17) and (18), without loss of generality, we can write

$$C_1 = \{(S, \gamma S), \gamma \in X_3\} \cup \{(N, \gamma N), \gamma \in X_4\}, \quad (20)$$

$$C_3 = \{(S, \gamma S), \gamma \in X_7\} \cup \{(N, \gamma N), \gamma \in X_8\}, \quad (21)$$

$$C_5 = \{(N, \gamma N), \gamma \in X_3\} \cup \{(S, \gamma S), \gamma \in X_4\}, \quad (22)$$

$$C_7 = \{(N, \gamma N), \gamma \in X_7\} \cup \{(S, \gamma S), \gamma \in X_8\}. \quad (23)$$

Obviously,

$$|X_3| + |X_4| = |X_7| + |X_8| = 2m + 1. \quad (24)$$

From [16] we know that

$$\sum_{i=1}^4 (|X_{2i-1}| - |X_{2i}|)^2 = 2(|X_3| - |X_4|)^2 + (|X_5| - |X_6|)^2 = q.$$

Therefore,

$$(|X_5| - |X_6|)^2 = x^2 \quad \text{and} \quad (|X_3| - |X_4|)^2 = (|X_7| - |X_8|)^2 = y^2 \quad (25)$$

for some x and y . Consequently, by (13), (19), (24) and (25), we have

$$\begin{aligned} \{|X_5|, |X_6|\} &= \left\{ m + \frac{1}{2}(1+x), m + \frac{1}{2}(1-x) \right\}, \\ \{|X_3|, |X_4|\} &= \{|X_7|, |X_8|\} = \left\{ m + \frac{1}{2}(1+y), m + \frac{1}{2}(1-y) \right\}. \end{aligned}$$

Since

$$\begin{aligned} &\{(S, \gamma S), \gamma \in X_1 \cup X_3 \cup X_5 \cup X_7\} \cup \{(N, \gamma N), \gamma \in X_2 \cup X_4 \cup X_6 \cup X_8\} \\ &= \bigcup_{i=1}^{2m} E_{8i} \cup \left(\bigcup_{i=1}^3 \bigcup_{j=0}^{2m} E_{8j+i} \right), \end{aligned}$$

it follows that

$$|X_1 \cup X_2 \cup X_3 \cup X_4 \cup X_5 \cup X_6 \cup X_7 \cup X_8| = 8m + 3,$$

i.e.

$$X_1 + X_2 + X_3 + X_4 + X_5 + X_6 + X_7 + X_8 = GF(q).$$

Now we are going to prove (6).

For any $h = \alpha w + \beta \neq 0, \alpha, \beta \in GF(q)$, it is clear that

$$\{hC_0, \dots, hC_7\} = \{C_0, \dots, C_7\}.$$

Note that

$$(\alpha, \beta)(\alpha', \beta') = (\alpha w + \beta)(\alpha' w + \beta') = (\alpha\beta' + \beta\alpha', \beta\beta' - \alpha\alpha'),$$

we have

$$\begin{aligned} hC_0 &= (\alpha S, \beta S) \cup \{(\alpha\gamma + \beta)S, (\beta\gamma - \alpha)S, \gamma \in X_1\} \\ &\quad \cup \{((\alpha\gamma + \beta)N, (\beta\gamma - \alpha)N), \gamma \in X_2\}, \\ hC_4 &= (\alpha N, \beta N) \cup \{(\alpha\gamma + \beta)N, (\beta\gamma - \alpha)N, \gamma \in X_1\} \\ &\quad \cup \{((\alpha\gamma + \beta)S, (\beta\gamma - \alpha)S), \gamma \in X_2\}. \end{aligned} \tag{26}$$

For any $\gamma_0 \in X_1$, we can choose $\alpha, \beta \in GF(q)$ such that $\alpha \in S$ and $\alpha^{-1}\beta = -\gamma_0 \in X_2$.

In (26) the term

$$(\alpha S, \beta S) = (S, -\gamma_0 S) \in C_4,$$

it follows that

$$hC_0 = C_4 \quad \text{and} \quad hC_4 = C_0.$$

Then in (26) the term

$$((\alpha\gamma_0 + \beta)S, (\beta\gamma_0 - \alpha)S) = (0, -(1 + \gamma_0^2)S)$$

should be equal to $(0, N)$, i.e., $1 + \gamma_0^2 \in S$. Now

$$\begin{aligned} hC_0 &= (S, -\gamma_0 S) \cup (0, N) \cup \{((\gamma - \gamma_0)S, -(1 + \gamma\gamma_0)S), \gamma \in X_1, \gamma \neq \gamma_0\} \\ &\quad \cup \{((\gamma - \gamma_0)N, -(1 + \gamma\gamma_0)N), \gamma \in X_2\} \\ &= (0, N) \cup (S, -\gamma_0 S) \cup \{(S, -\gamma^{-1}(1 + \gamma_0^2 + \gamma_0\gamma)S), \gamma \in R_1\} \\ &\quad \cup \{(N, -\gamma^{-1}(1 + \gamma_0^2 + \gamma_0\gamma)N), \gamma \in R_2\}, \end{aligned}$$

where

$$\begin{aligned} R_1 &= ((X_1 - \gamma_0) \cap S) \cup ((X_2 - \gamma_0) \cap N), \\ R_2 &= ((X_1 - \gamma_0) \cap N) \cup ((X_2 - \gamma_0) \cap S). \end{aligned}$$

Comparing expression (26) with (11), it follows that

$$|R_1| = |(X_1 - \gamma_0) \cap S| + |(X_2 - \gamma_0) \cap N| = |X_1| - 1, \tag{27}$$

$$|R_2| = |(X_1 - \gamma_0) \cap N| + |(X_2 - \gamma_0) \cap S| = |X_2|. \tag{28}$$

(27) and (28) mean that the coefficients of γ_0 in $X_1N + X_2S$ and $X_1S + X_2N$ are $|X_1| - 1$ and $|X_2|$ respectively.

Similarly, for $\gamma_0 \in X_1$, we can prove

$$hC_i = C_{i+4} \quad \text{and} \quad hC_{i+4} = C_i, \quad i = 1, 2, 3.$$

Comparing the expression of hC_i with that of C_{i+4} ($i = 1, 2, 3$), it follows that the coefficients of γ_0 in $X_3N + X_4S$, $X_3S + X_4N$, $X_5N + X_6S$, $X_5S + X_6N$, $X_7N + X_8S$ and $X_7S + X_8N$ are $|X_4|$, $|X_3|$, $|X_6|$, $|X_5|$, $|X_8|$ and $|X_7|$ respectively.

Similarly, repeating the procedure for X_2, \dots, X_8 , one can get (6). The theorem is proved. \square

For any subset $E \subset GF(q)$, $\beta, \gamma \in GF(q)$ and integer t , we write

$$\beta E + \gamma = \{\beta\alpha + \gamma : \alpha \in E\}, \quad E^{(t)} = \{\alpha^t : \alpha \in E\}$$

and as well as in $Z[GF(q)]$

$$\beta E + \gamma = \sum_{\alpha \in E} (\beta\alpha + \gamma), \quad E^{(t)} = \sum_{\alpha \in E} \alpha^t.$$

Theorem 2 Suppose $W = \{X_1, \dots, X_8\}$ is a $(q; x, 0, y, y)$ -partition of $GF(q)$, $\beta, \gamma \in GF(q)$ and $\beta \neq 0$. If $\bar{W} = \{\bar{X}_1, \dots, \bar{X}_8\}$ is obtained from W under the following transformations:

- (a) $\bar{X}_i = X_i + r$, $i = 1, \dots, 8$,
- (b) $\bar{X}_i = X_i^{(p)}$, $i = 1, \dots, 8$,
- (c) $\bar{X}_i = \beta X_i$, $i = 1, \dots, 8$ for $\beta \in S$,
- (d) $\bar{X}_1 = \beta X_2$, $\bar{X}_2 = \beta X_1$ and $\bar{X}_i = \beta X_i$, $i = 3, \dots, 8$ for $\beta \in N$,
then \bar{W} is also a $(q; x, 0, y, y)$ -partition of $GF(q)$.

The proof of Theorem 2 is trivial. We refer it to the reader.

Remark. In general, the representation $q = x^2 + 2y^2$ is not unique, and so the values of x and y in (1), (3) and (4) are not completely determined by Theorem 1. In this case there is a problem: Does there a $(q; x, 0, y, y)$ -partition exist for every given pair (x, y) satisfying (1)?

Example 1 $q = 27 = 8 \times 3 + 3 = 3^2 + 2 \times 3^2 = 5^2 + 2 \times 1^2$. Let δ be a root of the equation $\delta^3 = \delta + 2$. Then

$$GF(3^3)^* = \{\delta^i : i = 0, 1, \dots, 25\}.$$

Take

$$\begin{aligned} X_1 &= \{\delta^5, \delta^{15}, \delta^{19}\}, & X_2 &= \{\delta^2, \delta^6, \delta^{18}\}, \\ X_3 &= \{\delta^4, \delta^{10}, \delta^{12}, \delta^{13}\}, & X_4 &= \{\delta^{14}, \delta^{16}, \delta^{22}\}, \\ X_5 &= \{\delta^7, \delta^8, \delta^{11}, \delta^{20}, \delta^{21}, \delta^{24}\}, & X_6 &= \{0\}, \\ X_7 &= \{\delta, \delta^3, \delta^9\}, & X_8 &= \{\delta^0, \delta^{17}, \delta^{23}, \delta^{25}\}. \end{aligned}$$

It is easy to verify that $\{X_1, \dots, X_8\}$ is a $(27; 5, 0, 1, 1)$ -partition.

Remark. We can read off

$$|(X_{2i-1} - \alpha) \cap S| + |(X_{2i} - \alpha) \cap N|, \quad |(X_{2i-1} - \alpha) \cap N| + |(X_{2i} - \alpha) \cap S|$$

by simply finding the coefficients of $\alpha \in GF(q)$ in

$$X_{2i-1}N + X_{2i}S, \quad X_{2i-1}S + X_{2i}N$$

respectively, $i = 1, 2, 3, 4$.

3 Generalized cyclotomic classes

In this section, by using $(q; x, 0, y, y)$ -partitions, we will construct generalized cyclotomic classes, which have properties similar to those of classical cyclotomic classes.

For any $\alpha \in GF(q)^*$, we know that $\chi_\alpha(S)$ and $\chi_\alpha(N)$ only depend on the fact that α is in S or in N , and do not depend on the particular choice of the element α in S or N . If Q is either S or N , we will denote $\chi_\alpha(Q)$ by $\chi_S(Q)$ for any $\alpha \in S$ and $\chi_N(Q)$ for any $\beta \in N$. Define

$$a = \chi_S(S) = \chi_N(N), \quad b = \chi_S(N) = \chi_N(S).$$

The value of a and b can be computed from either the values of quadratic Gauss sums [6], [7], [8] or uniform cyclotomy [2]. They are

$$\{a, b\} = \left\{ -\frac{1}{2}(1 + \sqrt{-q}), -\frac{1}{2}(1 - \sqrt{-q}) \right\}.$$

Theorem 3 Suppose $\{X_i, i = 1, \dots, 8\}$ is a $(q; x, 0, y, y)$ -partition of $GF(q)$, and C_0, \dots, C_7 are subsets of $GF(q^2)$, given as in (10), (11), (17), (18), (20), (21), (22) and (23) respectively. Then

$$C_i C_j = \varepsilon_{j-i} (2m+1)(4m+1) + \sum_{k=0}^7 \langle j-i, k \rangle C_{i+k}, \quad 0 \leq i \leq j \leq 7, \quad (29)$$

where $\varepsilon_{j-i} = 1$ or 0 according as $j-i = 4$ or not, the table of $\langle i, j \rangle$ ($0 \leq i, j \leq 7$) reads as:

$$\begin{array}{cccccccc} \langle 0, 0 \rangle & \langle 0, 1 \rangle & \langle 0, 2 \rangle & \langle 0, 3 \rangle & \langle 0, 4 \rangle & \langle 0, 5 \rangle & \langle 0, 6 \rangle & \langle 0, 7 \rangle \\ \langle 1, 0 \rangle & \langle 1, 1 \rangle & \langle 1, 2 \rangle & \langle 1, 3 \rangle & \langle 1, 4 \rangle & \langle 1, 5 \rangle & \langle 1, 6 \rangle & \langle 1, 7 \rangle \\ \langle 2, 0 \rangle & \langle 2, 1 \rangle & \langle 2, 2 \rangle & \langle 2, 3 \rangle & \langle 2, 4 \rangle & \langle 2, 5 \rangle & \langle 2, 6 \rangle & \langle 2, 7 \rangle \\ \langle 3, 0 \rangle & \langle 3, 1 \rangle & \langle 3, 2 \rangle & \langle 3, 3 \rangle & \langle 3, 4 \rangle & \langle 3, 5 \rangle & \langle 3, 6 \rangle & \langle 3, 7 \rangle \\ \langle 4, 0 \rangle & \langle 4, 1 \rangle & \langle 4, 2 \rangle & \langle 4, 3 \rangle & \langle 4, 4 \rangle & \langle 4, 5 \rangle & \langle 4, 6 \rangle & \langle 4, 7 \rangle \\ \langle 5, 0 \rangle & \langle 5, 1 \rangle & \langle 5, 2 \rangle & \langle 5, 3 \rangle & \langle 5, 4 \rangle & \langle 5, 5 \rangle & \langle 5, 6 \rangle & \langle 5, 7 \rangle \\ \langle 6, 0 \rangle & \langle 6, 1 \rangle & \langle 6, 2 \rangle & \langle 6, 3 \rangle & \langle 6, 4 \rangle & \langle 6, 5 \rangle & \langle 6, 6 \rangle & \langle 6, 7 \rangle \\ \langle 7, 0 \rangle & \langle 7, 1 \rangle & \langle 7, 2 \rangle & \langle 7, 3 \rangle & \langle 7, 4 \rangle & \langle 7, 5 \rangle & \langle 7, 6 \rangle & \langle 7, 7 \rangle \end{array}$$

$$\begin{array}{ll} \langle 0, 0 \rangle = m^2 + m, & \langle 0, 1 \rangle = m^2 + m - \frac{1}{4}(y^2 - 1), \\ \langle 0, 2 \rangle = m^2 - m + \frac{1}{2}(y^2 - 1), & \langle 0, 4 \rangle = m^2 + 3m + 1, \\ \langle 1, 0 \rangle = m^2 + \frac{1}{4}(y^2 - 1), & \langle 1, 2 \rangle = m^2 + m + \frac{1}{4}(1 + 2xy + y^2), \\ \langle 1, 7 \rangle = m^2 + m + \frac{1}{4}(1 - 2xy + y^2), & \langle 2, 0 \rangle = m^2 + m - \frac{1}{2}(y^2 - 1), \end{array}$$

and $C_i = C_j$ as $i \equiv j \pmod{8}$.

Proof. We calculate the character values of $C_i, i = 0, \dots, 7$, as follows.

For any $\alpha_1, \alpha_2 \in GF(q)$, clearly,

$$\chi_{(\alpha_1, \alpha_2)}(C_{i+4}) = \overline{\chi_{(\alpha_1, \alpha_2)}(C_i)}, \quad i = 0, 1, 2, 3.$$

It is enough to calculate the character values only for C_0, C_1, C_2 and C_3 . Now

$$\chi_{(\alpha_1, \alpha_2)}(C_0) = \sum_{\beta \in S} \xi^{\text{Tr} q^2(\alpha_1 \beta w + \alpha_2 \beta)} + \sum_{\beta \in S, \gamma \in X_1} \xi^{\text{Tr} q^2((\alpha_1 \gamma + \alpha_2) \beta w + \alpha_2 \gamma \beta - \alpha_1 \beta)}$$

$$\begin{aligned}
& + \sum_{\beta \in N, \gamma \in X_2} \xi \text{Tr} q^2((\alpha_1 \gamma + \alpha_2) \beta w + \alpha_2 \gamma \beta - \alpha_1 \beta) \\
= & \sum_{\beta \in S} \xi \text{Tr} q(\text{Tr} q^2 / q(\alpha_1 \beta w + \alpha_2 \beta)) \\
& + \sum_{\beta \in S, \gamma \in X_1} \xi \text{Tr} q(\text{Tr} q^2 / q((\alpha_1 \gamma + \alpha_2) \beta w + \alpha_2 \gamma \beta - \alpha_1 \beta)) \\
& + \sum_{\beta \in N, \gamma \in X_2} \xi \text{Tr} q(\text{Tr} q^2 / q((\alpha_1 \gamma + \alpha_2) \beta w + \alpha_2 \gamma \beta - \alpha_1 \beta)) \\
= & \sum_{\beta \in S} \xi \text{Tr} q(2\alpha_2 \beta) + \sum_{\beta \in S, \gamma \in X_1} \xi \text{Tr} q(2(\alpha_2 \gamma - \alpha_1) \beta) + \sum_{\beta \in N, \gamma \in X_2} \xi \text{Tr} q(2(\alpha_2 \gamma - \alpha_1) \beta) \\
= & \chi_{2\alpha_2}(S) + \sum_{\gamma \in X_1} \chi_{2(\alpha_2 \gamma - \alpha_1)}(S) + \sum_{\gamma \in X_2} \chi_{2(\alpha_2 \gamma - \alpha_1)}(N).
\end{aligned}$$

If $\alpha_1 = \alpha_2 = 0$,

$$\chi_{(0,0)}(C_0) = |C_0| = (2m+1)(4m+1) = (q^2 - 1)/8.$$

If $\alpha_2 = 0, \alpha_1 \neq 0$,

$$\chi_{(\alpha_1,0)}(C_0) = 4m+1 + |X_1| \chi_{\alpha_1}(S) + |X_2| \chi_{\alpha_1}(N) = 3m+1.$$

If $\alpha_2 \in N$, we set $\alpha = \alpha_2^{-1} \alpha_1$, then

$$\begin{aligned}
\chi_{(\alpha_1, \alpha_2)}(C_0) &= a + \sum_{\gamma \in X_1} \chi_{\gamma - \alpha}(S) + \sum_{\gamma \in X_2} \chi_{\gamma - \alpha}(N) \\
&= a + \sum_{\gamma \in (X_1 - \alpha) \cap S} \chi_{\gamma}(S) + \sum_{\gamma \in (X_1 - \alpha) \cap N} \chi_{\gamma}(S) \\
&\quad + \sum_{\gamma \in (X_1 - \alpha) \cap \{0\}} \chi_{\gamma}(S) + \sum_{\gamma \in (X_2 - \alpha) \cap \{0\}} \chi_{\gamma}(N) \\
&\quad + \sum_{\gamma \in (X_2 - \alpha) \cap S} \chi_{\gamma}(N) + \sum_{\gamma \in (X_2 - \alpha) \cap N} \chi_{\gamma}(N) \\
&= (1 + k_1)a + k_2b + (4m+1)|((X_1 \cup X_2) - \alpha) \cap \{0\}|,
\end{aligned}$$

where

$$\begin{aligned}
k_1 &= |(X_1 - \alpha) \cap S| + |(X_2 - \alpha) \cap N|, \\
k_2 &= |(X_1 - \alpha) \cap N| + |(X_2 - \alpha) \cap S|.
\end{aligned}$$

If $\alpha_2 \in S$, let $\alpha = \alpha_2^{-1} \alpha_1$ again, we get

$$\chi_{(\alpha_1, \alpha_2)}(C_0) = (1 + k_1)b + k_2a + (4m+1)|((X_1 \cup X_2) - \alpha) \cap \{0\}|.$$

Similarly, we have

$$\begin{aligned}
\chi_{(0,0)}(C_i) &= (2m+1)(4m+1), \\
\chi_{(\alpha_1,0)}(C_i) &= |X_{2i+1}| \chi_{\alpha_1}(S) + |X_{2i+2}| \chi_{\alpha_1}(N), \\
\chi_{(\alpha_1, \alpha_2)}(C_i) &= \begin{cases} k_{2i+1}a + k_{2i+2}b + (4m+1)|((X_{2i+1} \cup X_{2i+2}) - \alpha) \cap \{0\}|, & \alpha_2 \in N, \\ k_{2i+1}b + k_{2i+2}a + (4m+1)|((X_{2i+1} \cup X_{2i+2}) - \alpha) \cap \{0\}|, & \alpha_2 \in S, \end{cases}
\end{aligned}$$

where

$$\begin{aligned} k_{2i+1} &= |(X_{2i+1} - \alpha) \cap S| + |(X_{2i+2} - \alpha \cap N|, \\ k_{2i+2} &= |(X_{2i+1} - \alpha) \cap N| + |(X_{2i+2} - \alpha \cap S|, \end{aligned}$$

$i = 1, 2, 3$.

We know that for any $\gamma \neq 0$

$$\begin{aligned} \chi_\gamma(S) + \chi_\gamma(N) &= -1, & \chi_\gamma(S)\chi_\gamma(N) &= 2m + 1, \\ \chi_\gamma^2(S) &= -2m - 1 - \chi_\gamma(S), & \chi_\gamma^2(N) &= -2m - 1 - \chi_\gamma(N). \end{aligned}$$

We denote the right hand side of (29) by R_{ij} and discuss the case $i = j = 0$ at first. One can see that

$$|C_0C_0| = (2m + 1)^2(4m + 1)^2$$

and

$$|R_{00}| = \sum_{k=0}^7 \langle 0, k \rangle |C_k| = (2m + 1)(4m + 1) \sum_{k=0}^7 \langle 0, k \rangle = |C_0C_0|.$$

For $\alpha_1 \neq 0$,

$$\chi_{(\alpha_1, 0)}(C_0C_0) = \chi_{(\alpha_1, 0)}^2(C_0) = (3m + 1)^2$$

and

$$\begin{aligned} \chi_{(\alpha_1, 0)}(R_{00}) &= (\langle 0, 0 \rangle + \langle 0, 4 \rangle)(3m + 1) - \langle 0, 2 \rangle (|X_3| + |X_4|) \\ &\quad - \langle 0, 1 \rangle (|X_5| + |X_6|) - \langle 0, 3 \rangle (|X_7| + |X_8|) \\ &= \chi_{(\alpha, 0)}(C_0C_0). \end{aligned}$$

For $\alpha_2 \in N$,

$$\begin{aligned} \chi_{(\alpha_1, \alpha_2)}(C_0C_0) &= -(2m + 1)(1 + k_1 - k_2)^2 - (1 + k_1)^2a - k_2^2b + \\ &\quad (4m + 1)[2(1 + k_1)a + 2k_2b + (4m + 1)]|((X_1 \cup X_2) - \alpha) \cap \{0\}| \end{aligned}$$

and

$$\begin{aligned} &\chi_{(\alpha_1, \alpha_2)}(R_{00}) \\ &= \langle 0, 0 \rangle [-1 - k_1 - k_2 + 2(4m + 1)]|((X_1 \cup X_2) - \alpha) \cap \{0\}| + \\ &\quad \langle 0, 1 \rangle [-k_3 - k_4 - k_7 - k_8 + 2(4m + 1)]|((X_3 \cup X_4 \cup X_7 \cup X_8) - \alpha) \cap \{0\}| \\ &\quad + \langle 0, 2 \rangle [-k_5 - k_6 + 2(4m + 1)]|((X_5 \cup X_6) - \alpha) \cap \{0\}| \\ &\quad + [\langle 0, 4 \rangle - \langle 0, 0 \rangle] \chi_{(\alpha_1, \alpha_2)}(C_4). \end{aligned}$$

If $\alpha = \alpha_2^{-1}\alpha_1 \in X_1 \cup X_2$, then

$$1 + k_1 = k_2 = m, \quad k_{2i+1} + k_{2i+2} = 2m + 1, \quad i = 1, 2, 3.$$

Hence,

$$\chi_{(\alpha_1, \alpha_2)}(C_0^2) = (3m + 1)^2 = \chi_{(\alpha_1, \alpha_2)}(R_{00}).$$

If $\alpha \in X_3$, then

$$\begin{aligned} k_1 &= |X_3| - 1, & k_2 &= |X_4|, & k_3 &= |X_1|, & k_4 &= |X_2|, \\ k_5 &= |X_8|, & k_6 &= |X_7|, & k_7 &= |X_5|, & k_8 &= |X_6|. \end{aligned}$$

Therefore

$$\chi_{(\alpha_1, \alpha_2)}(C_0^2) = -(2m+1)y^2 - |X_3|^2a - |X_4|^2b = \chi_{(\alpha_1, \alpha_2)}(R_{00}).$$

If $\alpha \in X_4$, then

$$\begin{aligned} k_1 &= |X_4| - 1, & k_2 &= |X_3|, & k_3 &= |X_2|, & k_4 &= |X_1|, \\ k_5 &= |X_7|, & k_6 &= |X_8|, & k_7 &= |X_6|, & k_8 &= |X_5|. \end{aligned}$$

So

$$\chi_{(\alpha_1, \alpha_2)}(C_0^2) = -(2m+1)y^2 - |X_4|^2a - |X_3|^2b = \chi_{(\alpha_1, \alpha_2)}(R_{00}).$$

By a similar discussion we can prove that

$$\chi_{(\alpha_1, \alpha_2)}(C_0^2) = \chi_{(\alpha_1, \alpha_2)}(R_{00})$$

is valid in all cases. Consequently, $C_0^2 = R_{00}$. The proof of the rest of the theorem is similar. \square

Theorem 3 shows that the formulas of the left part of the table [5, p196] are still valid for C_0, \dots, C_7 defined by (10), (11), (17), (18), (20), (21), (22), (23) respectively, which need not be cyclotomic sets. We call them generalized cyclotomic classes.

Corollary 1 *Under the same assumptions as Theorem 3, C_0, C_1, C_2 and C_3 are 4- $\{q^2; (q^2 - 1)/8; (q^2 - 9)/16\}$ SDS.*

Proof. From Theorem 3 we have

$$\sum_{i=0}^3 \Delta C_i = \sum_{i=0}^3 C_i C_{i+4} = (7q^2 + 1)/16 + (q^2 - 9)/16GF(q^2).$$

The proof is completed. \square

Remark. It is easy to see that C_i, C_j, C_k and C_l are 4- $\{q^2; (q^2 - 1)/8; (q^2 - 9)/16\}$ SDS for any set $\{i, j, k, l\} \equiv \{0, 1, 2, 3\} \pmod{4}$.

Example 2 *Let $q = 11$. Then $m = 1 = y, x = 3$.*

$$S = \{1, 3, 4, 5, 9\}, \quad N = \{2, 6, 7, 8, 10\}.$$

Take

$$\begin{aligned} X_1 &= \{6\}, & X_2 &= \{10\}, & X_3 &= \{0, 1\}, & X_4 &= \{9\}, \\ X_5 &= \emptyset, & X_6 &= \{2, 3, 8\}, & X_7 &= \{7\}, & X_8 &= \{4, 5\}. \end{aligned}$$

It is easy to verify that X_1, \dots, X_8 satisfy (2)-(9). Define C_0, \dots, C_7 as in (10), (11), (17), (18), (20)-(23):

$$\begin{aligned} C_0 &= (0, S) \cup (S, 6S) \cup (N, -N), & C_4 &= (0, N) \cup (N, 6N) \cup (S, -S), \\ C_1 &= (S, 0) \cup (S, S) \cup (N, 9N), & C_5 &= (N, 0) \cup (N, N) \cup (S, 9S), \\ C_2 &= (N, 2N) \cup (N, 3N) \cup (N, 8N), & C_6 &= (S, 2S) \cup (S, 3S) \cup (S, 8S), \\ C_3 &= (S, 7S) \cup (N, 4N) \cup (N, 5N), & C_7 &= (N, 7N) \cup (S, 4S) \cup (S, 5S). \end{aligned}$$

From Theorem 3 it follows that

$$\begin{aligned} \langle 0, 0 \rangle &= \langle 0, 1 \rangle = 2, & \langle 0, 2 \rangle &= 0, & \langle 0, 4 \rangle &= 5, \\ \langle 1, 0 \rangle &= \langle 1, 7 \rangle = 1, & \langle 1, 2 \rangle &= 4, & \langle 2, 0 \rangle &= 3. \end{aligned}$$

It is easy to show that C_0, \dots, C_7 satisfy (29). However they are generalized cyclotomic classes, not cyclotomic sets.

4 Constructing SDS

In this section we will construct some SDS which can be used to form Hadamard matrices.

To construct SDS in $GF(q^2)$ we need the following lemmas.

Lemma 1 *In $GF(q^2)$ the following equations hold:*

- (i) $\Delta E_i = (4m + 1) + 2m(E_i + E_{i+8m+4});$
- (ii) $\Delta(E_i, E_{i+8m+4}) = (4m + 1)(E_i + E_{i+8m+4});$
- (iii) $\Delta(E_i, E_j + E_{j+8m+4}) = GF(q^2)^* - (E_i + E_j + E_{i+8m+4} + E_{j+8m+4}),$
 $i \neq j, 0 \leq i, j \leq 16m + 7.$

For the proof see [13].

Let $A = \{a_0, \dots, a_{2t}\} \subset \{0, 1, \dots, 16m + 7\}$ and $B = \{b_1, \dots, b_{4m+1-t}\} \subset \{0, 1, \dots, 8m + 3\}$. Suppose

$$|\{a \pmod{8m + 4} : a \in A\} \cup B| = 4m + 2 + t. \quad (30)$$

Write

$$C = \bigcup_{i=0}^{2t} E_{a_i}, \quad \bar{C} = \bigcup_{i=0}^{2t} E_{a_i+8m+4}, \quad H = \bigcup_{j=1}^{4m+1-t} (E_{b_j} \cup E_{b_j+8m+4}), \quad D = C \cup H.$$

Lemma 2 *Under the condition (30) we have*

$$\begin{aligned} \Delta D &= 2(4m + 1 - t)(4m + 1) + [(4m + 1)^2 - t^2]GF(q^2)^* \\ &\quad - (4m + 1 - t)(C + \bar{C}) + \Delta C. \end{aligned} \quad (31)$$

Proof. (31) follows from Lemma 1 by direct calculation. \square

From (31) we see that the expression of ΔD only depends on the set of A and does not depend on the particular choice of B .

Let X_1, \dots, X_8 be a $(q; x, 0, y, y)$ -partition of $GF(q)$,

$$B_i \subset GF(q) \setminus (X_{2i+1} \cup X_{2i+2}), \quad |B_i| = 3m + 1, \quad i = 0, 1, 2, 3, \quad (32)$$

C_0, \dots, C_7 are given in Theorem 3. Set

$$H_i = \bigcup_{\gamma \in B_i} ((S, \gamma S) \cup (N, \gamma N)), \quad D_i = C_i \cup H_i, \quad i = 0, 1, 2, 3. \quad (33)$$

Theorem 4 D_0, D_1, D_2 and D_3 given in (33) are 4 - $\{q; q(q-1)/2; q(q-2)\}$ SDS .

Proof. It is easy to show that (32) and (33) guarantee the validity of (30) for every i .

From Lemma 2 we have

$$\Delta D_i = 2(3m + 1)(4m + 1) + (3m + 1)(5m + 1)GF(q^2) - (3m + 1)(C_i + C_{i+4}) + \Delta C_i,$$

$i = 0, 1, 2, 3$. The conclusion follows immediately from Theorem 3. \square

Let e, f be integers such that $0 < e, f < 7$ and $\{e, f\} \equiv \{1, 3\} \pmod{4}$,

$$B_i \subset X_{2(i+f)+1} \cup X_{2(i+f)+2}, \quad |B_i| = m, \quad (34)$$

$$H_i = \bigcup_{\gamma \in B_i} ((S, \gamma S) \cup (N, \gamma N)), \quad D_i = C_i \cup C_{i+2} \cup C_{i+e} \cup H_i, \quad (35)$$

$i = 0, 1, 2, 3$.

Theorem 5 D_0, D_1, D_2 and D_3 given in (34) and (35) are $4\text{-}\{q^2; q(q-1)/2, q(q-2)\}$ SDS.

Proof. First, (34) and (35) ensure (30) for every $D_i, i = 0, 1, 2, 3$.

Then, from Lemma 2 we have

$$\begin{aligned} \Delta D_i &= 2m(4m+1) + m(7m+2)GF(q^2)^* - m(GF(q^2)^* - C_{i+f} - C_{i+f+4}) \\ &\quad + \Delta(C_i + C_{i+2} + C_{i+e}). \end{aligned}$$

Hence

$$\begin{aligned} \sum_{i=0}^3 \Delta D_i &= 8m(4m+1) + m(28m+5)GF(q^2)^* + \sum_{i=0}^3 \Delta(C_i + C_{i+2} + C_{i+e}) \\ &= q^2 + q(q-2)GF(q^2), \end{aligned}$$

where we used the following equations:

$$\begin{aligned} \Delta(C_i + C_{i+2} + C_{i+e}) &= 3(2m+1)(4m+1) + \sum_{j=0}^3 \alpha_j (C_{i+j} + C_{i+j+4}), \\ \sum_{j=0}^3 \alpha_j &= 36m^2 + 27m + 3. \end{aligned}$$

□

Remark. The (1,-1) incidence matrices of D_0, D_1, D_2 and D_3 in Theorem 4 or Theorem 5 maybe used to construct an Hadamard matrix of order $4q^2$ with Goethals-Seidel or Wallis-Whiteman type [13].

5 Constructing T -matrices

T -matrices play an important role in composite Hadamard matrices.

Definition 1 (T -matrices) $(0, \pm 1)$ matrices T_1, T_2, T_3 and T_4 of order t are called T -matrices if the following 5 conditions are satisfied:

- (a) They are pairwise commute;
- (b) There is a monomial matrix R of order t , $R' = R$, $R^2 = I$, such that

$$(T_i R)' = T_i R, \quad i = 1, 2, 3, 4;$$

- (c) $T_i * T_j = 0, i \neq j, 1 \leq i, j \leq 4$, where $*$ denotes Hadamard product;

(d) $T_1 + T_2 + T_3 + T_4$ is a $(1, -1)$ matrix;

(e) $\sum_{i=1}^4 T_i T_i' = tI$.

Let C_0, \dots, C_7 be given as in (10), (11), (17), (18), (20)–(23) respectively. We know that for each i , $0 \leq i \leq 7$, there is a set A_i of numbers, such that

$$\begin{aligned} |A_i| &= 2m + 1, & A_i &\subset \{0, 1, \dots, 16m + 7\}, \\ C_i &= \bigcup_{j \in A_i} E_j, & i &= 0, 1, \dots, 7. \end{aligned}$$

It is clear that

$$\bigcup_{i=0}^7 A_i = \{0, 1, \dots, 16m + 7\},$$

and for $i = 0, 1, 2, 3$,

$$A_i + 8m + 4 = \{a + 8m + 4 \pmod{16m + 8} : a \in A_i\} = A_{i+4}.$$

For each i , $0 \leq i \leq 3$, choose a subset I_i of A_i such that $|I_i| = m$. Denote $A_i \setminus I_i$ by \bar{I}_i , $i = 0, 1, 2, 3$. Set

$$\begin{aligned} D_0 &= \bigcup_{i \in I_3} (E_i \cup E_{i+8m+4}) \cup C_0 \cup C_1 \cup C_2, \\ D_2 &= \bigcup_{i \in I_1} (E_i \cup E_{i+8m+4}) \cup C_2 \cup C_3 \cup C_4, \\ D_1 &= \bigcup_{i \in I_0} (E_i \cup E_{i+8m+4}) \cup \left(\bigcup_{j \in \bar{I}_1} (E_j \cup E_{j+8m+4}) \right) \\ &\quad \cup \left(\bigcup_{k \in I_2} (E_k \cup E_{k+8m+4}) \right) \cup C_3, \\ D_3 &= \bigcup_{i \in \bar{I}_0} (E_i \cup E_{i+8m+4}) \cup \left(\bigcup_{j \in I_2} (E_j \cup E_{j+8m+4}) \right) \\ &\quad \cup \left(\bigcup_{k \in I_3} (E_k \cup E_{k+8m+4}) \right) \cup C_5. \end{aligned} \tag{36}$$

Theorem 6 D_0, D_1, D_2 and D_3 defined in (36) are 4 - $\{q^2; q(q-1)/2; q(q-2)\}$ SDS.

Proof. First, we see that, for each D_i , $0 \leq i \leq 3$, (30) is valid. Then, from Lemma 2 it follows that

$$\begin{aligned} \Delta D_0 &= 2m(4m + 1) + m(7m + 1)GF(q^2)^* + m(C_3 + C_7) + \Delta(C_0 + C_1 + C_2), \\ \Delta D_2 &= 2m(4m + 1) + m(7m + 1)GF(q^2)^* + m(C_1 + C_5) + \Delta(C_2 + C_3 + C_4), \\ \Delta D_1 &= 2(3m + 1)(4m + 1) + (3m + 1)(5m + 1)GF(q^2)^* - \\ &\quad (3m + 1)(C_3 + C_7) + \Delta C_3, \\ \Delta D_3 &= 2(3m + 1)(4m + 1) + (3m + 1)(5m + 1)GF(q^2)^* - \\ &\quad (3m + 1)(C_1 + C_5) + \Delta C_5. \end{aligned}$$

From Theorem 3 it is easy to verify that

$$\sum_{i=0}^3 \Delta D_i = q^2 + q(q-2)GF(q^2).$$

The theorem is proved. □

We have need to point out that every element of $GF(q^2)$ appears an even number of times in the system of D_0, D_1, D_2 and D_3 given in (36). Hence from Theorem 1 and Theorem 3 of [14] and Theorem 6 above we obtain the following theorem.

Theorem 7 *There exist T-matrices of order q^2 with q prime power $\equiv 3 \pmod{8}$.*

It is worth pointing out that from [13], [15] and this paper one would know the state of the art concerning Hadamard matrices of order $4q^2$ (q prime power), namely, the only open case is q congruent $7 \pmod{8}$.

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