

On Orthogonal Designs in Order 48

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Abstract

We show that all 3164 possible $OD(48; s_1, s_2, s_3)$ exist. In addition to the use of some classical techniques we employ two new method of construction.

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1 Introduction

An *orthogonal design* A , of order n , and type (s_1, s_2, \dots, s_u) , denoted $OD(n; s_1, s_2, \dots, s_u)$ on the commuting variables $(\pm x_1, \pm x_2, \dots, \pm x_u, 0)$ is a square matrix of order n with entries $\pm x_k$ where each x_k occurs s_k times in each row and column such that the distinct rows are pairwise orthogonal.

In other words

$$AA^T = (s_1x_1^2 + \dots + s_ux_u^2)I_n$$

where I_n is the identity matrix. It is known that the maximum number of variables in an orthogonal design is $\rho(n)$, the Radon number, where for $n = 2^ab$, b odd, set $a = 4c + d$, $0 \leq d < 4$, then $\rho(n) = 8c + 2^d$.

Section 2 is devoted to the classical results. By classical results we mean all orthogonal designs obtained more than 25 years ago. To show the evolution of the subject area, we keep the new results out of this section. By new results we mean results obtained in the new century.

2 Classical Results

For this part we make extensive use of the book of Geramita and Seberry [4]. For convenience we quote the following theorems we use:

Theorem 1 (Equating and Killing Theorem [4, Lemma 4.4]) *If A is an orthogonal design $OD(n; s_1, s_2, \dots, s_u)$ on the commuting variables $(\pm x_1, \pm x_2, \dots, \pm x_u, 0)$ then there is an orthogonal design $OD(n; s_1, s_2, \dots, s_i + s_j, \dots, s_u)$ and $OD(n; s_1, s_2, \dots, s_{j-1}, s_{j+1}, \dots, s_u)$ on the $u - 1$ commuting variables $(\pm x_1, \pm x_2, \dots, \pm x_{j-1}, \pm x_{j+1}, \dots, \pm x_u, 0)$.*

We systematically use orthogonal designs and amicable orthogonal designs to construct orthogonal designs in order 48. Explicitly

Theorem 2 (Multiplication Theorem [4, Lemma 4.11]) *If there exists an orthogonal design $OD(n; s_1, s_2, \dots, s_u)$ then there exists an orthogonal design $OD(2n; s_1, s_1, es_2, \dots, es_u)$ where $e = 1$ or 2 .*

Theorem 3 *We use the following designs in order 24 and the Multiplication Theorem, Theorem 2, to obtain the designs given in the Appendix which have 9, 8, and 7 variables in order 48.*

8 variables	Ref	7 Variables	Ref	6 Variables	Ref
1 1 1 1 1 1 1 1	I	1 1 1 1 1 4 5	IX	1 1 1 2 4 10	XI
1 1 1 1 1 1 1 4	II	1 1 1 1 2 2 8	X	1 1 1 3 4 9	XII
1 1 1 1 1 1 9 9	III			1 1 1 4 6 6	XIII
1 1 1 1 1 2 8 9	IV			1 1 2 3 4 9	XIV
1 1 1 1 1 4 4 4	V				
1 1 1 1 1 5 5 9	VI				
1 1 1 1 4 4 4 4	VII				
3 3 3 3 3 3 3 3	VIII				

Table 1: Order 24

We note that the designs in order 24 given in Table 1 may be found in [4, p373-375]: the $OD(24; 3, 3, 3, 3, 3, 3, 3, 3)$ was found by Plotkin and all the other 8-variable designs by Peter J. Robinson. We have included in Table 2 all the 7-variable designs that can be made from the 8- variable designs and then all the 6- variable designs that can be made from all the 7-variables designs using the Equating and Killing Theorem 1 in order 24.

Remark 1 Those designs in the Appendix marked by a capital roman numeral followed by a number, eg $V-2$, are designs which exist in order 24 and hence in order 48.

7 Variables	Ref	7 Variables	Ref	7 Variables	Ref
1 1 1 1 1 1 1	I-1	1 1 1 1 1 5 5	VI-1	1 1 1 1 4 4 5	V-3
1 1 1 1 1 1 2	I-2	1 1 1 1 1 5 9	VI-2	1 1 1 1 4 4 8	VII-2
1 1 1 1 1 1 4	II-1	1 1 1 1 1 5 14	VI-3	1 1 1 1 5 5 9	VI-4
1 1 1 1 1 1 5	II-2	1 1 1 1 1 8 9	IV-4	1 1 1 1 5 5 10	VI-5
1 1 1 1 1 1 9	III-1	1 1 1 1 1 8 11	IV-5	1 1 1 1 5 6 9	VI-6
1 1 1 1 1 1 18	III-2	1 1 1 1 1 9 9	III-3	1 1 1 2 2 8 9	IV-8
1 1 1 1 1 2 4	II-3	1 1 1 1 1 9 10	III-4	1 1 1 2 4 4 4	V-4
1 1 1 1 1 2 8	IV-1	1 1 1 1 2 2 8	X	1 1 1 2 5 5 9	VI-7
1 1 1 1 1 2 9	IV-2	1 1 1 1 2 8 9	IV-6	1 1 1 4 4 4 4	VII-3
1 1 1 1 1 2 17	IV-3	1 1 1 1 2 8 10	IV-7	1 1 1 4 4 4 5	VII-4
1 1 1 1 1 4 4	V-1	1 1 1 1 2 9 9	III-5	1 1 2 4 4 4 4	VII-5
1 1 1 1 1 4 5	IX	1 1 1 1 3 8 9	IV-7	3 3 3 3 3 3 3	VIII-1
1 1 1 1 1 4 8	V-2	1 1 1 1 4 4 4	VII-1	3 3 3 3 3 3 6	VIII-2

Table 2: 7-Variables in Orders 24 and 48.

6-Variables	Ref	6-Variables	Ref	6-Variables	Ref
1 1 1 1 1 1	I-1-1	1 1 1 1 5 15	VI-3-2	1 1 1 4 5 8	VII-2-5
1 1 1 1 1 2	I-1-2	1 1 1 1 6 9	VI-2-4	1 1 1 4 6 6	XIII
1 1 1 1 1 3	I-2-1	1 1 1 1 6 14	VI-3-3	1 1 1 4 8 9	IV-7-2
1 1 1 1 1 4	II-1-1	1 1 1 1 8 8	VII-2-2	1 1 1 5 5 9	VI-4-1
1 1 1 1 1 5	II-1-2	1 1 1 1 8 9	IV-4-3	1 1 1 5 5 10	VI-4-2
1 1 1 1 1 6	II-2-1	1 1 1 1 8 10	IV-4-1	1 1 1 5 5 11	VI-5-1
1 1 1 1 1 8	IV-1-2	1 1 1 1 8 11	IV-5-1	1 1 1 5 6 9	VI-4-3
1 1 1 1 1 9	III-1-1	1 1 1 1 8 12	IV-5-2	1 1 1 5 6 10	VI-5-2
1 1 1 1 1 10	III-1-2	1 1 1 1 9 9	III-4-1	1 1 1 5 7 9	VI-6-1
1 1 1 1 1 11	V-2-1	1 1 1 1 9 10	III-3-1	1 1 1 6 6 9	VI-6-2
1 1 1 1 1 12	V-2-2	1 1 1 1 9 11	III-3-2	1 1 2 2 2 8	X-5
1 1 1 1 1 14	VI-2-2	1 1 1 1 9 17	III-4-2	1 1 2 2 8 9	IV-8-1
1 1 1 1 1 17	IV-3-1	1 1 1 1 10 10	III-3-3	1 1 2 2 8 10	IV-6-3
1 1 1 1 1 18	III-2-1	1 1 1 2 2 4	II-3-2	1 1 2 2 9 9	III-3-3
1 1 1 1 1 19	III-2-2	1 1 1 2 2 8	IV-1-4	1 1 2 3 4 9	XIV
1 1 1 1 2 2	I-1-2	1 1 1 2 2 9	IV-2-4	1 1 2 3 8 9	IV-7-3
1 1 1 1 2 4	II-1-3	1 1 1 2 2 17	IV-3-4	1 1 2 4 4 4	V-4-2
1 1 1 1 2 5	II-2-2	1 1 1 2 3 8	X-4	1 1 2 4 4 5	V-3-4
1 1 1 1 2 8	IV-1-1	1 1 1 2 4 4	V-1-3	1 1 2 4 4 8	VII-2-6
1 1 1 1 2 9	III-1-3	1 1 1 2 4 5	IX-4	1 1 2 5 5 9	VI-4-4
1 1 1 1 2 10	IV-2-2	1 1 1 2 4 8	V-2-5	1 1 2 5 5 10	VI-5-3
1 1 1 1 2 17	IV-3-2	1 1 1 2 4 10	XI	1 1 2 5 6 9	VI-6-3
1 1 1 1 2 18	III-2-3	1 1 1 2 5 5	VI-1-3	1 1 3 4 4 4	V-4-3
1 1 1 1 3 4	II-3-1	1 1 1 2 5 9	VI-2-5	1 1 3 5 5 9	VI-7-1
1 1 1 1 3 8	IV-1-3	1 1 1 2 5 14	VI-3-4	1 1 4 4 4 4	VII-3-1
1 1 1 1 3 9	IV-2-4	1 1 1 2 8 9	IV-4-2	1 1 4 4 4 5	VII-4-1
1 1 1 1 3 17	IV-3-3	1 1 1 2 8 10	IV-6-1	1 1 4 4 4 6	VII-4-4
1 1 1 1 4 4	V-1-1	1 1 1 2 8 11	IV-5-3	1 1 4 4 5 5	VII-4-2
1 1 1 1 4 5	V-1-2	1 1 1 2 9 10	III-3-4	1 2 2 2 8 9	IV-8-2
1 1 1 1 4 6	IX-3	1 1 1 3 4 9	XII	1 2 2 4 4 4	V-4-4
1 1 1 1 4 8	V-2-1	1 1 1 3 8 9	IV-7-1	1 2 2 5 5 9	VI-7-2
1 1 1 1 4 9	V-2-3	1 1 1 3 8 10	IV-6-2	1 2 4 4 4 4	VII-3-2
1 1 1 1 4 12	VII-2-3	1 1 1 3 9 9	III-3-2	1 2 4 4 4 5	VII-4-3
1 1 1 1 5 5	VI-1-1	1 1 1 4 4 4	V-4-1	1 3 4 4 4 4	VII-5-1
1 1 1 1 5 6	VI-1-2	1 1 1 4 4 5	V-3-1	2 2 4 4 4 4	VII-5-2
1 1 1 1 5 8	V-2-4	1 1 1 4 4 6	V-3-2	3 3 3 3 3 3	VIII-1-1
1 1 1 1 5 9	VI-2-1	1 1 1 4 4 8	VII-2-1	3 3 3 3 3 6	VIII-1-2
1 1 1 1 5 10	VI-2-3	1 1 1 4 4 9	VII-2-4	3 3 3 3 3 9	VIII-1-3
1 1 1 1 5 14	VI-3-1	1 1 1 4 5 5	V-3-3	3 3 3 3 6 6	VIII-1-4

Table 3: 6-Variable designs in orders 24 and 48

Theorem 4 (Second Multiplication Theorem [4, Lemma 4.11]) *If there exists an orthogonal design $OD(n; s_1, s_2, \dots, s_u)$ then there exists an orthogonal design $OD(2n; e_1s_1, e_2s_2, \dots, e_us_u)$ where $e_i = 1$ or 2 , $i = 1, \dots, u$.*

Theorem 5 (Construction 1) *An $OD(6; a, b)$ and an $AOD(8; (a_1, a_2, \dots, a_s); (b_1, b_2, \dots, b_t))$ give an $OD(48; aa_1, aa_2, \dots, aa_s, bb_1, bb_2, \dots, bb_t)$.*

Remark 2 The designs in the Appendix marked Th2 have been made using the Multiplication Theorem with the designs in Tables 1 and 2.

The designs in the Appendix marked Th4 have been made using the Second Multiplication Theorem, Theorem 4, with the designs in Table 1.

Table 4 gives designs that can be constructed in order 48 using Theorem 5.5.

□

Theorem 6

(Construction 2) An $OD(12; a, b, c, d)$ and an $AOD(4; (a_1, a_2, a_3, a_4); (\beta))$ give an $OD(48; aa_1, aa_2, aa_3, aa_4, \beta b, \beta c, \beta d)$.

Remark 3 This theorem does not give any designs in 48 not able to be constructed by other methods.

Theorem 7 (Construction 3) An $OD(12; a, b)$ and an $AOD(4; (a_1, a_2, \dots, a_s); (b_1, b_2, \dots, b_t))$ give an $OD(48; aa_1, aa_2, \dots, aa_s, bb_1, bb_2, \dots, bb_t)$.

Corollary 1 Since every orthogonal design $OD(12; a, b)$ exists $1 \leq a, b, a + b \leq 12$ except $(a, b) = (1, 7), (3, 5)$ and $(4, 7)$ and since $AOD(4; (1, 1, 1, 1), (1))$, $AOD(4; (1), (1, 1, 1, 1))$, $AOD(4; (1, 1, 1), (1, 1, 1))$, and $AOD(4; (1, 1, 2), (1, 1, 2))$, exist we have the existence of all $OD(48; a, a, a, a, b)$, $OD(48; a, a, a, b, b, b)$ and $OD(48; a, a, 2a, b, b, 2b)$ for the given a, b .

Remark 4 In the Appendix the label $12\beta\gamma$ indicates the construction method using Construction 3. $\beta = a, b, \dots, y, z, A, B, \dots, G$ according as the $OD(12; a, b)$ has $(a, b) = (1, 1), (1, 2), (1, 3), (1, 4), (1, 5), (1, 6), (1, 8), (1, 9), (1, 10), (1, 11), (2, 2), (2, 3), (2, 4), (2, 5), (2, 6), (2, 7), (2, 8), (2, 9), (2, 10), (3, 3), (3, 4), (3, 6), (3, 7), (3, 8), (3, 9), (4, 4), (4, 5), (4, 6), (4, 8), (5, 5), (5, 6), (5, 7), (6, 6)$.

$\gamma = a, b, c, d$ according as the AOD is $AOD(4; (1), (1, 1, 1, 1))$, $AOD(4; (1, 1, 1, 1), (1))$, $AOD(4; (1, 1, 1), (1, 1, 1))$ or $AOD(4; (1, 1, 2), (1, 1, 2))$.

Label	New Design	Weight	$OD(6; a, b)$	$AOD(8; (a_1, a_2, a_3, a_4), (b_1, b_2, b_3, b_4))$
Th5a	(1 1 2 2 2 4 4)	16	(1, 1)	(1, 1, 2, 4);(2, 2, 4)
Th5b	(1 1 2 2 2 32)	40	(1, 4)	(1, 1, 2, 2, 2);(8)
Th5c	(1 1 2 4 8 8 16)	40	(1, 4)	(1, 1, 2, 4);(2, 2, 4)
Th5d	(1 1 7 7)	16	(1, 1)	(1, 7);(1, 7)
Th5e	(1 2 2 2 3 6)	16	(1, 1)	(1, 2, 2, 3);(2, 6)
Th5f	(1 2 2 3 8 24)	24	(1, 4)	(1, 2, 2, 3);(2, 6)
Th5g	(1 4 7 28)	40	(1, 4)	(1, 7);(1, 7)
Th5h	(2 2 4 4 4 8 16)	40	(4, 1)	(1, 1, 2, 4);(2, 2, 4)
Th5i	(2 4 4 4 6 12)	32	(2, 2)	(1, 2, 2, 3);(2, 6)
Th5j	(2 4 6 8 8 12)	40	(4, 1)	(1, 2, 2, 3);(2, 6)
Th5k	(4 4 8 8 8 8)	40	(4, 1)	(1, 1, 2, 2, 2);(8)

Table 4: Designs in 48 made using Theorem 5.

Theorem 8 (Construction 4, Robinson [4, p267]) A product design of type $PD(48; (1, 1, 1; 1, 1, 1; 9))$ and an $AOD(4; (u_1, u_2, \dots, u_s); (v, w_1, \dots, w_t))$ give one of

- $OD(48; v, v, v, w, w, w, 9u_1, 9u_2, \dots, 9u_s)$;
- $OD(48; v, v, v, bw_1, bw_2, \dots, bw_t, 9u_1, 9u_2, \dots, 9u_s)$, $b = 1, 2$ or 3 .

Theorem 9 (Construction 5) *An $OD(16; a_1, a_2, \dots, a_s)$ and suitable 3×3 matrices then there are orthogonal designs on s or fewer variables in order 48.*

Remark 5 Theorem 8 gives the designs marked Th8, and Theorem 9 gives the designs marked Th9 in the Appendix.

Table 5 gives the designs in order 16 we use to construct designs indicated by Th9 in the Appendix. □

9 Variables	Ref	8 Variables	Ref
1 1 1 1 1 1 1 1 1	16a	1 1 1 1 1 1 3 3	16j
1 1 1 1 1 1 1 1 2	16b	1 1 1 1 1 1 4 4	16k
1 1 1 1 1 1 1 1 4	16c	1 1 1 1 1 1 5 5	16l
1 1 1 1 1 1 1 1 8	16d	1 1 1 1 2 2 3 3	16m
1 1 1 1 1 1 2 2 2	16e	1 1 1 1 2 2 4 4	16n
1 1 1 1 1 2 2 2 2	16f	1 1 2 2 2 2 3 3	16o
1 1 1 1 1 2 3 3 3	16g		
1 1 1 1 2 2 2 2 2	16h		
1 1 2 2 2 2 2 2 2	16i		
8 Variables Derived by Th1	Ref	8 Variables Derived by Th1	Ref
1 1 1 1 1 1 1 1	16a1	1 1 1 1 1 3 3 5	16g3
1 1 1 1 1 1 1 2	16a2	1 1 1 1 2 2 2 2	16e4
1 1 1 1 1 1 1 3	16b1	1 1 1 1 2 2 2 3	16f2
1 1 1 1 1 1 1 4	16c1	1 1 1 1 2 2 2 4	16h1
1 1 1 1 1 1 1 5	16c2	1 1 1 1 2 3 3 3	16g4
1 1 1 1 1 1 1 8	16d1	1 1 1 1 2 3 3 4	16g5
1 1 1 1 1 1 1 9	16d2	1 1 1 1 3 3 3 3	16g6
1 1 1 1 1 1 2 2	16b2	1 1 1 2 2 2 2 2	16f4
1 1 1 1 1 1 2 4	16c3	1 1 1 2 2 2 2 3	16h2
1 1 1 1 1 1 2 8	16d3	1 1 1 2 2 3 3 3	16g7
1 1 1 1 1 2 2 2	16e1	1 1 2 2 2 2 2 2	16i1
1 1 1 1 1 2 2 3	16e2	1 1 2 2 2 2 2 4	16i2
1 1 1 1 1 2 2 4	16f3	1 2 2 2 2 2 2 2	16i3
1 1 1 1 1 2 3 3	16g1	1 2 2 2 2 2 2 3	16i4
1 1 1 1 1 2 3 6	16g2	2 2 2 2 2 2 2 2	16i5
1 1 1 1 1 3 3 3	16g3		

Table 5: Designs in order 16 and 48.

Theorem 10 (Construction 6) *If A is an $OD(24; a_1, a_2, \dots, a_s)$ then*

$$D = \begin{pmatrix} I & A \\ -A^T & I \end{pmatrix}$$

where A^T means the transpose of A , is an $OD(48; 1, a_1, a_2, \dots, a_s)$.

Theorem 11 (Construction 7) *If A is an $OD(12; a_1, a_2, \dots, a_s)$ then*

$$D = \begin{pmatrix} I & A & B & C \\ -A & I & C & -B \\ -B & -C & I & A \\ -C & B & -A & I \end{pmatrix}$$

where A^T means the transpose of A , then $OD(48; 1, a_1, a_2, \dots, a_s)$, $OD(48; 1, 2a_1, 2a_2, \dots, 2a_s)$, and $OD(48; 1, 3a_1, 3a_3, \dots, 3a_s)$ also exist.

Theorem 12 (Construction 8) *If $A = OD(12; a_1, a_2, \dots, a_s)$ is then replacing the variable of an $OD(6; 5)$ by A gives an $OD(48; 5a_1, 5a_2, \dots, 5a_s)$. If A is symmetric then*

$$D = \begin{pmatrix} 0 & A & A & xI & A & -A \\ A & 0 & A & A & -A & xI \\ A & A & 0 & -A & xI & A \\ -xI & -A & A & 0 & A & A \\ -A & A & -xI & A & 0 & A \\ A & -xI & -A & A & A & 0 \end{pmatrix}$$

and if A is skew symmetric then

$$D = \begin{pmatrix} xI & A & A & 0 & A & -A \\ A & xI & A & A & -A & 0 \\ A & A & xI & -A & 0 & A \\ 0 & -A & A & xI & A & A \\ -A & A & 0 & A & xI & A \\ A & 0 & -A & A & A & xI \end{pmatrix}$$

give an $OD(48; 1, 4a_1, 4a_2, \dots, 4a_s)$.

Corollary 2 *The $OD(8; 1, 1, 1, 1, 1, 1, 1, 1)$ gives an $OD(48; 5, 5, 5, 5, 5, 5, 5, 5)$. The $OD(8; 1, 1, 1, 1, 1, 1, 1, 1)$ may be made skew symmetric and used to form the $OD(48; 1, 4, 4, 4, 4, 4, 4, 4)$.*

8-rm Variables	Ref
1 4 4 4 4 4 4 4	Th12
5 5 5 5 5 5 5 5	Th12

Table 6: Designs constructed using Construction 8.

Theorem 13 (Construction 9) *All $OD(24; s_1, s_2, 24 - s_1 - s_2)$ exist so all $OD(48; t_1, t_2, 48 - t_1 - t_2)$ exist.*

Theorem 14 *All $OD(48; s_1, s_2, s_3)$ $0 \leq s_1, s_2, s_3, s_1 + s_2 + s_3 \leq 48$ can be constructed using classical results except possibly the following 18 types:*

1	5	41	7	7	33	7	17	23	15	15	15	13	17	17	7	7	27
5	11	31	7	11	29	7	15	25	7	13	21	11	17	17	13	17	17
5	5	37	5	17	25	7	13	25	5	13	23	7	11	23	9	17	17

3 Construction using 16 circulant matrices

Following [10], a set $\{A_1, A_2, \dots, A_{2n}\}$ of square real matrices is said to be amicable if

$$\sum_{i=1}^n (A_{\sigma(2i-1)} A_{\sigma(2i)}^t - A_{\sigma(2i)} A_{\sigma(2i-1)}^t) = 0$$

for some permutation σ of the set $\{1, 2, \dots, 2n\}$.

For simplicity, we will always take $\sigma(i) = i$ unless otherwise specified. Clearly a set of mutually amicable matrices is amicable, but the converse is not true in general.

Theorem 15 [10, Theorem 4 and Example 6] *Let $\{A_i\}_{i=1}^8$ be an amicable set of circulant matrices such that $\sum_{i=1}^4 (A_{2i-1} A_{2i-1}^t + 3A_{2i} A_{2i}^t)$ is a multiple of the identity matrix. Then the matrix*

$$H = \begin{pmatrix} A_1 & A_2 & A_2 & A_2 & -A_3 R & A_4 R & A_4 R & A_4 R \\ -A_2 & A_1 & -A_2 & A_2 & A_4 R & A_4 R & -A_4 R & A_3 R \\ -A_2 & A_2 & A_1 & -A_2 & A_4 R & -A_4 R & A_3 R & A_4 R \\ -A_2 & -A_2 & A_2 & A_1 & A_4 R & A_3 R & A_4 R & -A_4 R \\ A_3 R & -A_4 R & -A_4 R & -A_4 R & A_1 & A_2 & A_2 & A_2 \\ -A_4 R & -A_4 R & A_4 R & -A_3 R & -A_2 & A_1 & -A_2 & A_2 \\ -A_4 R & A_4 R & -A_3 R & -A_4 R & -A_2 & A_2 & A_1 & -A_2 \\ -A_4 R & -A_3 R & -A_4 R & A_4 R & -A_2 & -A_2 & A_2 & A_1 \\ A_5 R & -A_6 R & -A_6 R & -A_6 R & A_7^t R & A_8^t R & A_8^t R & A_8^t R \\ -A_6 R & -A_6 R & A_6 R & -A_5 R & A_8^t R & A_8^t R & -A_8^t R & -A_7^t R \\ -A_6 R & A_6 R & -A_5 R & -A_6 R & A_8^t R & -A_8^t R & -A_7^t R & A_8^t R \\ -A_6 R & -A_5 R & -A_6 R & A_6 R & A_8^t R & -A_7^t R & A_8^t R & -A_8^t R \\ A_7 R & -A_8 R & -A_8 R & -A_8 R & -A_5^t R & -A_6^t R & -A_6^t R & -A_6^t R \\ -A_8 R & -A_8 R & A_8 R & -A_7 R & -A_6^t R & -A_6^t R & A_6^t R & A_5^t R \\ -A_8 R & A_8 R & -A_7 R & -A_8 R & -A_6^t R & A_6^t R & A_5^t R & -A_6^t R \\ -A_8 R & -A_7 R & -A_8 R & A_8 R & -A_6^t R & A_5^t R & -A_6^t R & A_6^t R \\ -A_5 R & A_6 R & A_6 R & A_6 R & -A_7 R & A_8 R & A_8 R & A_8 R \\ A_6 R & A_6 R & -A_6 R & A_5 R & A_8 R & A_8 R & -A_8 R & A_7 R \\ A_6 R & -A_6 R & A_5 R & A_6 R & A_8 R & -A_8 R & A_7 R & A_8 R \\ A_6 R & A_5 R & A_6 R & -A_6 R & A_8 R & A_7 R & A_8 R & -A_8 R \\ -A_7^t R & -A_8^t R & -A_8^t R & -A_8^t R & A_5^t R & A_6^t R & A_6^t R & A_6^t R \\ -A_8^t R & -A_8^t R & A_8^t R & A_7^t R & A_6^t R & A_6^t R & -A_6^t R & -A_5^t R \\ -A_8^t R & A_8^t R & A_7^t R & -A_8^t R & A_6^t R & -A_6^t R & -A_5^t R & A_6^t R \\ -A_8^t R & A_7^t R & -A_8^t R & A_8^t R & A_6^t R & -A_5^t R & A_6^t R & -A_6^t R \\ A_1 & A_2 & A_2 & A_2 & -A_3^t R & -A_4^t R & -A_4^t R & -A_4^t R \\ -A_2 & A_1 & -A_2 & A_2 & -A_4^t R & -A_4^t R & A_4^t R & A_3^t R \\ -A_2 & A_2 & A_1 & -A_2 & -A_4^t R & A_4^t R & A_3^t R & -A_4^t R \\ -A_2 & -A_2 & A_2 & A_1 & -A_4^t R & A_3^t R & -A_4^t R & A_4^t R \\ A_3^t R & A_4^t R & A_4^t R & A_4^t R & A_1 & A_2 & A_2 & A_2 \\ A_4^t R & A_4^t R & -A_4^t R & -A_3^t R & -A_2 & A_1 & -A_2 & A_2 \\ A_4^t R & -A_4^t R & -A_3^t R & A_4^t R & -A_2 & A_2 & A_1 & -A_2 \\ A_4^t R & -A_3^t R & A_4^t R & -A_4^t R & -A_2 & -A_2 & A_2 & A_1 \end{pmatrix},$$

is an orthogonal matrix, where R is the back-diagonal identity matrix of an appropriate order.

We now apply an algorithm developed in [6, 7, 8] which uses a known set of circulant matrices of order 3 of type (s_1, s_2, s_3, s_4) to construct an amicable set of eight matrices suitable for the array H in 15.

Start with a set of four circulant matrices A, B, C and D in variables a, b, c and d of type (s_1, s_2, s_3, s_4) , for which $AA^t + BB^t + CC^t + DD^t = (s_1a^2 + s_2b^2 + s_3c^2 + s_4d^2)I_3$. From matrices A, B, C and D construct four new circulant matrices, E, F, G and H by replacing a by e, b by f, c by g and d by h . The new matrices obviously satisfy an additive property as above but in variables e, f, g and h . We then searched to find a pairing, while reducing the number of variables to four by collapsing different variables, between the sets $\{A, B, C, D\}$ and $\{E, F, G, H\}$ to form an amicable set of eight matrices. All but two of the amicable sets listed in Table 8 were obtained from either the second or the third of the sequences listed in Table 7.

The amicable sets for $(1, 7, 7, 33)$ and $(1, 7, 17, 23)$ required a refinement of the above approach. Instead of constructing E, F, G and H from A, B, C and D , we selected E, F, G and H to be another set of four circulant matrices obtained from those in Table 7 and in addition allowed for circular permutation of the entries. In the case of $(1, 7, 7, 33)$ a matching was found between the matrices obtained from the $(1, 1, 1, 9)$ line in Table 7 and a permute of $(2, 2, 4, 4)$ while for $(1, 7, 17, 23)$ a matching was found between the matrices for $(1, 1, 5, 5)$ and $(1, 2, 3, 6)$.

To obtain amicable sets for $(5, 13, 23)$ and $(7, 13, 21)$ as listed in Table 9, the above approach was extended by considering known non-full OD's. Specifically, these two sets were obtained from the third, respectively the first, of the sets listed in Table 7 after zeroing out one of the variables, matched with the set $(a, c, -c), (b, d, -d), (c, c, 0), (d, d, 0)$ of type $(1, 1, 4, 4)$. Other combinations did not generate any new types.

Amicable sets of the above type, where A, B, C and D , are matched with E, F, G and H are called special amicable sets (see [7]).

Theorem 16 *All of the 18 orthogonal designs of order 48 in the Theorem 14, except possibly those of the following types are constructible from the array H above:*

$$\begin{array}{cccccc} 11 & 17 & 17 & 7 & 11 & 23 & 9 & 17 & 17 \\ 13 & 17 & 17 & 7 & 7 & 27 & & & \end{array}$$

Remark 6 *As noted in [7, Theorem 3], special amicable sets can be used in all arrays obtained by Kharaghani in [10]. Consequently each give infinitely many new orthogonal designs.*

4 Construction using 4 negacyclic matrices

Having exhausted all possible avenues and still having five orthogonal designs unresolved forced us to look to alternatives. Obviously order 48 is too large for a non-structured search. Our previous experience, see [9], had proven that negacyclic matrices are much more versatile for searches in the case where the order of block matrices is even. A negacyclic matrix of order n is any polynomial in the negashift matrix U of order n , where $U = [u_{ij}], u_{i(i+1)} = 1, i = 1, 2, \dots, n-$

type	A	B	C	D
(1, 1, 1, 9)	$(a, d, -d)$	$(b, d, -d)$	$(c, d, -d)$	(d, d, d)
(1, 1, 2, 8)	$(a, d, -d)$	$(b, d, -d)$	(c, d, d)	$(-c, d, d)$
(1, 1, 5, 5)	$(a, c, -c)$	$(b, d, -d)$	(c, d, d)	$(-d, c, c)$
(1, 2, 3, 6)	$(a, d, -d)$	(c, d, d)	$(c, -d, b)$	$(c, -b, -d)$
(2, 2, 4, 4)	(a, c, d)	$(a, -d, -c)$	$(b, c, -d)$	$(b, -c, d)$
(3, 3, 3, 3)	(a, b, c)	$(-b, a, d)$	$(-c, -d, a)$	$(-d, c, -b)$

Table 7: Known [1,pg. 348] full OD's of order 12 in 4 variables

type	A_1	A_3	A_5	A_7
	A_2	A_4	A_6	A_8
(1, 1, 5, 41)	(a, d, \bar{d})	(b, d, \bar{d})	(c, d, d)	(\bar{c}, d, d)
	(d, d, d)	(\bar{d}, d, d)	(c, d, \bar{d})	(d, d, \bar{d})
(1, 5, 5, 37)	(a, c, \bar{c})	(d, b, \bar{b})	(c, b, b)	(\bar{b}, c, c)
	(d, d, d)	(\bar{d}, d, d)	(d, d, \bar{d})	(d, d, \bar{d})
(1, 5, 11, 31)	(a, c, \bar{c})	(d, c, \bar{c})	(b, c, c)	(\bar{b}, c, c)
	(d, d, d)	(\bar{d}, d, d)	(b, d, \bar{d})	(c, d, \bar{d})
(1, 5, 17, 25)	(a, c, \bar{c})	(d, c, \bar{c})	(b, c, c)	(\bar{b}, c, c)
	(c, d, d)	(\bar{c}, d, d)	(b, d, \bar{d})	(c, d, \bar{d})
(1, 7, 7, 33)	(\bar{b}, \bar{d}, d)	(a, d, \bar{d})	(c, d, \bar{d})	(d, d, d)
	(\bar{d}, c, d)	(b, d, d)	(\bar{d}, b, \bar{d})	(c, \bar{d}, d)
(1, 7, 11, 29)	(b, c, \bar{c})	(a, c, \bar{c})	(d, c, c)	(\bar{d}, c, c)
	(b, d, d)	(\bar{b}, d, d)	(d, d, \bar{d})	(c, d, \bar{d})
(3, 7, 13, 25)	(c, c, c)	(\bar{c}, c, c)	(b, c, \bar{c})	(d, c, \bar{c})
	(a, d, \bar{d})	(c, d, \bar{d})	(b, d, d)	(\bar{b}, d, d)
(1, 7, 15, 25)	(a, d, \bar{d})	(b, d, \bar{d})	(d, d, d)	(\bar{d}, d, d)
	(d, c, c)	(\bar{c}, d, d)	(b, d, \bar{d})	(b, c, \bar{c})
(1, 7, 17, 23)	(a, c, \bar{c})	(b, d, \bar{d})	(c, d, d)	(\bar{d}, c, c)
	(c, d, d)	(c, d, \bar{d})	(\bar{d}, c, \bar{b})	(\bar{d}, b, c)
(3, 15, 15, 15)	(b, b, b)	(\bar{b}, b, b)	(b, b, \bar{b})	(b, b, \bar{b})
	(b, c, \bar{c})	(a, d, \bar{d})	(c, d, d)	(\bar{d}, c, c)

Table 8: Amicable sets for some full OD's of order 48 in 4 variables

type	A_1	A_3	A_5	A_7
	A_2	A_4	A_6	A_8
(5, 13, 23)	(b, a, \bar{a})	(a, a, \bar{c})	(a, c, c)	$(c, \bar{c}, 0)$
	$(0, c, c)$	$(\bar{b}, \bar{b}, 0)$	(\bar{c}, \bar{c}, c)	(b, \bar{b}, c)
(7, 13, 21)	(a, c, \bar{c})	(b, c, \bar{c})	(c, c, c)	$(c, \bar{c}, 0)$
	$(0, b, b)$	(\bar{a}, \bar{c}, c)	$(\bar{c}, \bar{c}, 0)$	(b, \bar{b}, a)

Table 9: Amicable sets for some OD's of order 48 in 3 variables

1, $u_{n1} = -1$, and zero otherwise. We searched for four negacyclic matrices A_i , $i = 1, 2, 3, 4$ of order 12 in three variables satisfying a suitable additive property.

As is known, we then substituted them in the Goethals-Seidel array

$$G = \begin{pmatrix} A_1 & A_2R & A_3R & A_4R \\ -A_2R & A_1 & A_4^tR & -A_3^tR \\ -A_3R & -A_4^tR & A_1 & A_2^tR \\ -A_4R & A_3^tR & -A_2^tR & A_1 \end{pmatrix},$$

to get the remaining orthogonal designs. Table 10 shows the first row of each of A_i , $i = 1, 2, 3, 4$. Note that the two orthogonal designs of types $(5, 13, 23)$ and $(7, 13, 21)$ were also obtained using four negacyclic matrices of order 12.

Remark 7 There are 73056 possible 8-tuples, 58844 possible 7-tuples, 41024 possible 6-tuples, 23532 possible 5-tuples, 10359 possible 4-tuples, 3164 possible 3-tuples, and 575 possible 2-tuples, before theory is applied to eliminate cases, which might give orthogonal designs in order 48.

Corollary 3 *All orthogonal designs of order 48 in three variables exist.*

type	A_1 A_2 A_3 A_4
(5, 13, 23)	$(\bar{c}, \bar{c}, \bar{c}, \bar{c}, \bar{c}, \bar{c}, c, c, \bar{c}, \bar{c}, \bar{c})$ $(\bar{c}, \bar{c}, c, \bar{c}, c, \bar{c}, c, \bar{c}, \bar{c}, c, 0, \bar{c})$ $(\bar{a}, \bar{b}, \bar{a}, b, b, b, 0, \bar{b}, b, b, \bar{a}, b)$ $(\bar{a}, 0, \bar{b}, 0, \bar{a}, \bar{b}, b, 0, 0, 0, b, \bar{b})$
(7, 11, 23)	$(\bar{c}, \bar{c}, \bar{c}, \bar{c}, \bar{c}, \bar{c}, c, c, \bar{c}, \bar{c}, \bar{c})$ $(\bar{b}, \bar{c}, c, c, \bar{c}, c, \bar{c}, c, \bar{c}, c, c, \bar{c})$ $(\bar{a}, \bar{b}, \bar{a}, b, \bar{a}, b, 0, \bar{b}, a, \bar{b}, \bar{a}, \bar{b})$ $(\bar{a}, \bar{b}, 0, \bar{b}, 0, 0, \bar{a}, \bar{b}, 0, b, 0, 0)$
(7, 13, 21)	$(\bar{c}, \bar{c}, \bar{c}, \bar{c}, \bar{c}, \bar{c}, c, c, \bar{c}, \bar{c}, \bar{c})$ $(\bar{a}, \bar{c}, \bar{a}, 0, 0, \bar{b}, \bar{b}, \bar{c}, c, \bar{b}, \bar{a}, 0)$ $(\bar{a}, \bar{b}, 0, 0, b, c, \bar{a}, 0, \bar{b}, b, c, \bar{c})$ $(\bar{a}, b, 0, b, \bar{a}, c, \bar{b}, \bar{b}, \bar{c}, b, \bar{b}, c)$
(7, 7, 27)	$(\bar{c}, \bar{c}, \bar{c}, \bar{c}, \bar{c}, \bar{c}, c, c, \bar{c}, \bar{c}, \bar{c})$ $(\bar{a}, \bar{a}, \bar{c}, \bar{b}, \bar{c}, c, \bar{c}, 0, b, \bar{c}, b, \bar{a})$ $(\bar{a}, c, \bar{a}, 0, b, b, \bar{b}, \bar{a}, \bar{c}, c, 0, 0)$ $(\bar{a}, \bar{c}, c, \bar{b}, \bar{c}, 0, c, \bar{c}, c, 0, 0, \bar{c})$
(9, 17, 17)	$(\bar{b}, \bar{c}, \bar{c}, \bar{c}, \bar{c}, c, c, \bar{c}, \bar{c}, c, \bar{c}, c)$ $(\bar{b}, \bar{b}, \bar{c}, \bar{b}, 0, c, \bar{c}, \bar{c}, b, \bar{c}, 0, \bar{b})$ $(\bar{b}, \bar{b}, \bar{b}, \bar{b}, c, \bar{b}, b, \bar{b}, b, b, \bar{b}, \bar{b})$ $(\bar{a}, \bar{a}, \bar{a}, 0, a, \bar{a}, 0, \bar{a}, \bar{a}, 0, a, \bar{a})$
(11, 17, 17)	$(\bar{b}, \bar{c}, \bar{c}, \bar{c}, \bar{c}, c, c, \bar{c}, \bar{c}, c, \bar{c}, c)$ $(\bar{b}, \bar{b}, \bar{c}, \bar{b}, 0, c, \bar{c}, \bar{c}, b, \bar{c}, 0, \bar{b})$ $(\bar{b}, \bar{b}, \bar{b}, \bar{b}, c, \bar{b}, b, \bar{b}, b, b, \bar{b}, \bar{b})$ $(\bar{a}, \bar{a}, \bar{a}, \bar{a}, 0, \bar{a}, a, \bar{a}, a, a, \bar{a}, \bar{a})$
(13, 17, 17)	$(0, \bar{a}, \bar{a}, \bar{b}, \bar{c}, \bar{a}, a, \bar{a}, \bar{c}, \bar{b}, \bar{a}, \bar{a})$ $(\bar{a}, b, \bar{a}, c, b, \bar{b}, \bar{a}, c, b, b, \bar{a}, \bar{c})$ $(\bar{a}, c, \bar{b}, \bar{b}, c, \bar{c}, \bar{b}, c, c, \bar{b}, \bar{b}, \bar{c})$ $(\bar{a}, \bar{c}, \bar{b}, b, \bar{c}, c, \bar{b}, \bar{c}, \bar{c}, b, \bar{b}, c)$

Table 10: First rows of four negacyclic matrices of order 12 in 3 variables

5 Appendix

9 – Variables	Ref	9 – Variables	Ref
1 1 1 1 1 1 1 1 1	Th2, I	1 1 1 1 2 3 3 3 9	Th9
1 1 1 1 1 1 1 1 2	16b	1 1 1 1 4 4 4 4 4	Th2, VII
1 1 1 1 1 1 1 1 4	Th2, II	1 1 1 2 2 2 2 2 4	Th9, 16h
1 1 1 1 1 1 1 1 8	16d	1 1 1 2 2 2 2 2 8	16f
1 1 1 1 1 1 1 2 4	Th9, 16b	1 1 1 2 2 2 9 9 9	Th8
1 1 1 1 1 1 1 4 4	Th2, II	1 1 1 2 2 2 9 9 18	Th8
1 1 1 1 1 1 1 4 8	Th9, 16d	1 1 1 2 2 3 3 3 8	16g
1 1 1 1 1 1 1 9 9	Th2, III	1 1 1 3 3 3 9 9 9	Th8
1 1 1 1 1 1 2 2 2	16e	1 1 1 3 3 3 9 9 18	Th8
1 1 1 1 1 1 2 2 8	Th9, 16e	1 1 2 2 2 2 2 2 2	Th2, I
1 1 1 1 1 1 2 8 9	Th2, IV	1 1 2 2 2 2 2 2 8	Th2, II
1 1 1 1 1 1 4 4 4	Th2, V	1 1 2 2 2 2 2 18 18	Th2, III
1 1 1 1 1 1 5 5 9	Th2, VI	1 1 2 2 2 2 4 16 18	Th2, IV
1 1 1 1 1 1 9 9 9	Th2, III	1 1 2 2 2 2 8 8 8	Th2, V
1 1 1 1 1 1 9 9 18	Th8	1 1 2 2 2 2 10 10 18	Th2, VI
1 1 1 1 1 2 2 2 2	16f	1 1 2 2 2 8 8 8 8	Th2, VII
1 1 1 1 1 2 2 2 4	Th9, 16e	1 2 2 2 2 2 2 2 4	Th9, 16i
1 1 1 1 1 2 2 2 8	Th9, 16f	1 3 3 3 3 3 3 3 3	Th10, VIII
1 1 1 1 1 2 2 8 9	Th2, IV	2 2 2 2 2 2 2 2 8	16i
1 1 1 1 1 2 3 3 3	16g	2 2 2 2 2 2 2 4 4	Th2, II
1 1 1 1 1 2 3 3 12	Th9, 16g	2 2 2 2 2 2 2 16 18	Th2, IV
1 1 1 1 1 2 8 8 9	Th2, IV	2 2 2 2 2 2 9 9 18	Th2, III
1 1 1 1 1 2 8 9 9	Th2, IV	2 2 2 2 2 4 4 8 8	Th2, V
1 1 1 1 1 4 4 4 4	Th2, V	2 2 2 2 2 4 8 8 18	Th2, IV
1 1 1 1 1 5 5 5 9	Th2, VI	2 2 2 2 2 4 9 9 16	Th2, IV
1 1 1 1 1 5 5 9 9	Th2, VI	2 2 2 2 2 5 5 10 18	Th2, VI
1 1 1 1 2 2 2 2 2	16h	2 2 2 2 2 9 9 10 10	Th2, VI
1 1 1 1 2 2 2 2 4	Th9, 16f	2 2 2 2 4 4 8 8 8	Th2, VII
1 1 1 1 2 2 2 2 8	Th9, 16g	3 3 3 3 3 3 3 3 3	Th2, VIII
1 1 1 1 2 3 3 3 4	Th9, 16h	3 3 6 6 6 6 6 6 6	Th2, VIII

Table 11: 9-Variable designs in order 48

8 – Variables	Ref	8 – Variables	Ref
1 1 1 1 1 1 1 1	Thm1	1 1 1 1 1 1 9 10	Thm1
1 1 1 1 1 1 1 2	Thm1	1 1 1 1 1 1 9 18	Thm1
1 1 1 1 1 1 1 3	Thm1	1 1 1 1 1 1 9 27	Thm1
1 1 1 1 1 1 1 4	Thm1	1 1 1 1 1 1 18 18	Thm1
1 1 1 1 1 1 1 5	Thm1	1 1 1 1 1 2 2 2	Thm1
1 1 1 1 1 1 1 6	Thm1	1 1 1 1 1 2 2 3	Thm1
1 1 1 1 1 1 1 8	Thm1	1 1 1 1 1 2 2 4	Thm1
1 1 1 1 1 1 1 9	Thm1	1 1 1 1 1 2 2 6	Thm1
1 1 1 1 1 1 1 11	Thm9	1 1 1 1 1 2 2 8	Thm1
1 1 1 1 1 1 1 12	Thm1	1 1 1 1 1 2 2 9	Thm1
1 1 1 1 1 1 1 18	Thm1	1 1 1 1 1 2 2 10	Thm1
1 1 1 1 1 1 2 2	Thm1	1 1 1 1 1 2 2 11	Thm9
1 1 1 1 1 1 2 4	Thm1	1 1 1 1 1 2 2 17	Thm1
1 1 1 1 1 1 2 5	Thm1	1 1 1 1 1 2 3 3	Thm1
1 1 1 1 1 1 2 8	Thm1	1 1 1 1 1 2 3 6	Thm1
1 1 1 1 1 1 2 9	Thm1	1 1 1 1 1 2 3 6	Thm1
1 1 1 1 1 1 2 10	Thm1	1 1 1 1 1 2 3 8	Thm1
1 1 1 1 1 1 2 17	Thm1	1 1 1 1 1 2 3 11	Thm1
1 1 1 1 1 1 3 3	Tab5	1 1 1 1 1 2 3 12	Thm1
1 1 1 1 1 1 3 4	Thm1	1 1 1 1 1 2 3 14	Thm1
1 1 1 1 1 1 3 9	Thm9	1 1 1 1 1 2 3 15	Thm1
1 1 1 1 1 1 3 12	Thm9	1 1 1 1 1 2 4 4	Thm1
1 1 1 1 1 1 4 4	Thm1	1 1 1 1 1 2 4 5	Thm9
1 1 1 1 1 1 4 5	Thm1	1 1 1 1 1 2 4 8	Thm1
1 1 1 1 1 1 4 8	Thm1	1 1 1 1 1 2 4 9	Thm9
1 1 1 1 1 1 4 9	Thm1	1 1 1 1 1 2 4 18	Thm9
1 1 1 1 1 1 4 16	Thm9	1 1 1 1 1 2 5 6	Thm9
1 1 1 1 1 1 5 5	Thm1	1 1 1 1 1 2 6 11	Thm1
1 1 1 1 1 1 5 6	Thm9	1 1 1 1 1 2 6 12	Thm1
1 1 1 1 1 1 5 8	Thm1	1 1 1 1 1 2 8 8	Thm1
1 1 1 1 1 1 5 9	Thm1	1 1 1 1 1 2 8 10	Thm9
1 1 1 1 1 1 5 14	Thm1	1 1 1 1 1 2 8 9	Thm1
1 1 1 1 1 1 5 20	Thm9	1 1 1 1 1 2 8 10	Thm1
1 1 1 1 1 1 8 9	Thm1	1 1 1 1 1 2 8 11	Thm1
1 1 1 1 1 1 8 11	Thm1	1 1 1 1 1 2 8 16	Thm4
1 1 1 1 1 1 9 9	Thm1	1 1 1 1 1 2 8 17	Thm1

Table 12: 8-Variable designs in order 48

8 – Variables	Ref	8 – Variables	Ref	8 – Variables	Ref
1 1 1 1 1 2 8 18	Thm1	1 1 1 1 2 2 2 5	Thm1	1 1 1 1 3 3 4 5	Thm1
1 1 1 1 1 2 9 9	Thm1	1 1 1 1 2 2 2 6	Thm1	1 1 1 1 3 3 5 9	Thm9
1 1 1 1 1 2 9 10	Thm1	1 1 1 1 2 2 2 8	Thm1	1 1 1 1 3 8 8 9	Thm1
1 1 1 1 1 2 9 16	Thm1	1 1 1 1 2 2 2 9	Thm1	1 1 1 1 3 8 9 9	Thm1
1 1 1 1 1 2 9 17	Thm1	1 1 1 1 2 2 2 10	Thm1	1 1 1 1 4 4 4 4	Thm1
1 1 1 1 1 2 9 18	Thm4	1 1 1 1 2 2 2 11	Thm9	1 1 1 1 4 4 4 5	Thm1
1 1 1 1 1 2 10 12	Thm9	1 1 1 1 2 2 3 3	Tab5	1 1 1 1 4 4 4 8	Thm1
1 1 1 1 1 3 3 3	Thm1	1 1 1 1 2 2 3 4	Thm1	1 1 1 1 4 4 4 16	Thm9
1 1 1 1 1 3 3 4	Thm9	1 1 1 1 2 2 3 8	Thm1	1 1 1 1 5 5 5 9	Thm1
1 1 1 1 1 3 3 5	Thm1	1 1 1 1 2 2 3 9	Thm9	1 1 1 1 5 5 5 10	Thm1
1 1 1 1 1 3 3 9	Thm9	1 1 1 1 2 2 3 12	Thm9	1 1 1 1 5 5 6 9	Thm1
1 1 1 1 1 3 3 11	Thm1	1 1 1 1 2 2 4 4	Thm1	1 1 1 1 5 5 9 9	Thm1
1 1 1 1 1 3 3 12	Thm1	1 1 1 1 2 2 4 8	Thm1	1 1 1 1 5 5 9 10	Thm1
1 1 1 1 1 3 3 13	Thm1	1 1 1 1 2 2 4 18	Thm9	1 1 1 1 5 6 9 9	Thm1
1 1 1 1 1 3 3 14	Thm1	1 1 1 1 2 2 5 8	Thm9	1 1 1 2 2 2 2 2	Thm1
1 1 1 1 1 3 5 11	Thm1	1 1 1 1 2 2 8 8	Thm9	1 1 1 2 2 2 2 3	Thm1
1 1 1 1 1 3 5 12	Thm1	1 1 1 1 2 2 8 9	Thm1	1 1 1 2 2 2 2 4	Thm1
1 1 1 1 1 3 8 9	Thm1	1 1 1 1 2 2 8 10	Thm1	1 1 1 2 2 2 2 5	Thm1
1 1 1 1 1 4 4 4	Thm1	1 1 1 1 2 2 9 9	Thm1	1 1 1 2 2 2 2 6	Thm1
1 1 1 1 1 4 4 5	Thm1	1 1 1 1 2 2 10 12	Thm9	1 1 1 2 2 2 2 8	Thm1
1 1 1 1 1 4 4 8	Thm1	1 1 1 1 2 3 3 3	Thm1	1 1 1 2 2 2 2 9	Thm1
1 1 1 1 1 4 5 5	Thm9	1 1 1 1 2 3 3 4	Thm1	1 1 1 2 2 2 2 10	Thm1
1 1 1 1 1 4 5 6	Thm9	1 1 1 1 2 3 3 7	Thm1	1 1 1 2 2 2 2 11	Thm9
1 1 1 1 1 4 5 8	Thm9	1 1 1 1 2 3 3 8	Thm9	1 1 1 2 2 2 3 4	Thm1
1 1 1 1 1 4 8 8	Thm4	1 1 1 1 2 3 3 9	Thm9	1 1 1 2 2 2 3 8	Thm1
1 1 1 1 1 4 8 9	Thm1	1 1 1 1 2 3 3 11	Thm1	1 1 1 2 2 2 3 9	Thm9
1 1 1 1 1 4 9 16	Thm4	1 1 1 1 2 3 3 12	Thm1	1 1 1 2 2 2 4 4	Thm1
1 1 1 1 1 5 5 5	Thm1	1 1 1 1 2 3 3 13	Thm1	1 1 1 2 2 2 4 5	Thm9
1 1 1 1 1 5 5 9	Thm1	1 1 1 1 2 3 4 6	Thm1	1 1 1 2 2 2 4 8	Thm9
1 1 1 1 1 5 5 10	Thm1	1 1 1 1 2 3 4 11	Thm1	1 1 1 2 2 2 5 6	Thm9
1 1 1 1 1 5 5 14	Thm1	1 1 1 1 2 3 4 12	Thm1	1 1 1 2 2 2 8 9	Thm1
1 1 1 1 1 5 5 18	Thm1	1 1 1 1 2 3 6 9	Thm9	1 1 1 2 2 2 8 10	Thm9
1 1 1 1 1 5 6 8	Thm9	1 1 1 1 2 3 8 9	Thm1	1 1 1 2 2 2 9 9	Thm1
1 1 1 1 1 5 6 9	Thm1	1 1 1 1 2 3 9 15	Thm9	1 1 1 2 2 2 9 18	Thm1
1 1 1 1 1 5 9 9	Thm1	1 1 1 1 2 4 4 4	Thm1	1 1 1 2 2 2 9 27	Thm1
1 1 1 1 1 5 9 10	Thm1	1 1 1 1 2 4 4 8	Thm9	1 1 1 2 2 2 18 18	Thm1
1 1 1 1 1 5 9 14	Thm1	1 1 1 1 2 4 4 16	Thm9	1 1 1 2 2 3 3 3	Thm1
1 1 1 1 1 8 8 9	Thm1	1 1 1 1 2 5 5 9	Thm1	1 1 1 2 2 3 3 4	Thm9
1 1 1 1 1 8 8 11	Thm1	1 1 1 1 2 5 9 10	Thm4	1 1 1 2 2 3 3 8	Thm9
1 1 1 1 1 8 9 9	Thm1	1 1 1 1 2 8 8 9	Thm1	1 1 1 2 2 3 3 9	Thm9
1 1 1 1 1 8 9 10	Thm1	1 1 1 1 2 8 8 10	Thm1	1 1 1 2 2 3 3 11	Thm1
1 1 1 1 1 8 9 11	Thm1	1 1 1 1 2 8 9 9	Thm1	1 1 1 2 2 3 3 12	Thm1
1 1 1 1 1 9 9 9	Thm1	1 1 1 1 2 8 9 10	Thm1	1 1 1 2 2 3 6 8	Thm9
1 1 1 1 1 9 9 10	Thm1	1 1 1 1 2 9 9 9	Thm1	1 1 1 2 2 3 8 15	Thm9
1 1 1 1 1 9 9 18	Thm1	1 1 1 1 2 9 9 18	Thm1	1 1 1 2 2 4 4 4	Thm4
1 1 1 1 1 9 9 19	Thm1	1 1 1 1 3 3 3 3	Thm1	1 1 1 2 2 4 4 8	Thm4
1 1 1 1 1 9 10 10	Thm4	1 1 1 1 3 3 3 4	Thm1	1 1 1 2 2 4 8 8	Thm4
1 1 1 1 1 9 10 18	Thm1	1 1 1 1 3 3 3 6	Thm1	1 1 1 2 2 4 8 9	Thm4
1 1 1 1 2 2 2 2	Thm1	1 1 1 1 3 3 3 9	Thm9	1 1 1 2 2 4 9 16	Thm4
1 1 1 1 2 2 2 3	Thm1	1 1 1 1 3 3 3 11	Thm1	1 1 1 2 2 5 5 9	Thm4
1 1 1 1 2 2 2 4	Thm1	1 1 1 1 3 3 3 12	Thm1	1 1 1 2 2 5 5 10	Thm4

Table 12(cont): 8-Variable designs in order 48

8 – Variables	Ref	8 – Variables	Ref	8 – Variables	Ref
1 1 1 2 2 5 9 10	Thm4	1 1 2 2 2 2 3 8	Thm9	1 1 2 2 3 3 3 4	Thm1
1 1 1 2 2 8 8 9	Thm1	1 1 2 2 2 2 3 9	Thm9	1 1 2 2 3 3 3 8	Thm9
1 1 1 2 2 8 9 9	Thm1	1 1 2 2 2 2 3 12	Thm9	1 1 2 2 3 3 3 9	Thm9
1 1 1 2 2 9 9 9	Thm1	1 1 2 2 2 2 4 5	Thm9	1 1 2 2 3 3 4 8	Thm9
1 1 1 2 2 9 9 11	Thm1	1 1 2 2 2 2 4 8	Thm1	1 1 2 2 3 3 8 12	Thm9
1 1 1 2 2 9 9 18	Thm1	1 1 2 2 2 2 4 16	Thm1	1 1 2 2 3 9 9 9	Thm1
1 1 1 2 2 9 9 20	Thm1	1 1 2 2 2 2 4 18	Thm1	1 1 2 2 3 9 9 18	Thm1
1 1 1 2 2 9 10 10	Thm4	1 1 2 2 2 2 4 34	Thm1	1 1 2 2 4 4 4 4	Thm4
1 1 1 2 2 9 11 18	Thm1	1 1 2 2 2 2 5 6	Thm9	1 1 2 2 4 4 8 8	Thm4
1 1 1 2 3 3 3 4	Thm1	1 1 2 2 2 2 5 9	Thm9	1 1 2 2 4 4 16 18	Thm1
1 1 1 2 3 3 3 5	Thm1	1 1 2 2 2 2 8 8	Thm1	1 1 2 2 4 8 8 8	Thm1
1 1 1 2 3 3 5 8	Thm1	1 1 2 2 2 2 8 9	Thm4	1 1 2 2 4 10 10 18	Thm1
1 1 1 2 3 3 3 8	Thm9	1 1 2 2 2 2 8 10	Thm2	1 1 2 2 8 8 8 8	Thm1
1 1 1 2 3 3 3 9	Thm9	1 1 2 2 2 2 8 16	Thm1	1 1 2 2 8 8 8 10	Thm1
1 1 1 2 3 3 3 10	Thm9	1 1 2 2 2 2 9 9	Thm4	1 1 2 3 3 3 3 8	Thm1
1 1 1 2 3 3 4 4	Thm1	1 1 2 2 2 2 9 16	Thm4	1 1 2 3 3 3 4 5	Thm9
1 1 1 2 3 3 4 9	Thm9	1 1 2 2 2 2 9 18	Thm4	1 1 2 3 3 3 5 6	Thm9
1 1 1 2 3 3 9 12	Thm9	1 1 2 2 2 2 10 10	Thm1	1 1 2 3 3 3 5 8	Thm9
1 1 1 2 4 4 4 4	Thm1	1 1 2 2 2 2 10 12	Thm9	1 1 2 3 3 3 5 9	Thm9
1 1 1 2 4 4 4 8	Thm4	1 1 2 2 2 2 10 18	Thm1	1 1 2 3 3 3 9 10	Thm9
1 1 1 2 4 4 8 8	Thm4	1 1 2 2 2 2 10 28	Thm1	1 1 2 4 4 4 4 4	Thm1
1 1 1 2 4 8 8 8	Thm4	1 1 2 2 2 2 16 18	Thm1	1 1 2 4 8 8 8 8	Thm1
1 1 1 2 4 9 9 9	Thm1	1 1 2 2 2 2 16 22	Thm1	1 1 3 3 3 9 9 9	Thm1
1 1 1 2 4 9 9 18	Thm1	1 1 2 2 2 2 18 18	Thm1	1 1 3 3 3 9 9 10	Thm1
1 1 1 2 5 5 5 9	Thm1	1 1 2 2 2 2 18 20	Thm1	1 1 3 3 3 9 9 18	Thm1
1 1 1 2 5 5 9 9	Thm1	1 1 2 2 2 3 3 8	Thm9	1 1 3 3 3 9 9 19	Thm1
1 1 1 3 3 3 3 4	Thm1	1 1 2 2 2 4 4 4	Thm4	1 1 3 3 3 9 10 18	Thm1
1 1 1 3 3 3 4 8	Thm1	1 1 2 2 2 4 4 8	Thm4	1 1 3 3 4 9 9 9	Thm1
1 1 1 3 3 3 3 9	Thm9	1 1 2 2 2 4 4 16	Thm2	1 1 3 3 4 9 9 18	Thm1
1 1 1 3 3 3 9 9	Thm1	1 1 2 2 2 4 8 8	Thm4	1 2 2 2 2 2 2 2	Thm1
1 1 1 3 3 3 9 18	Thm1	1 1 2 2 2 4 8 9	Thm4	1 2 2 2 2 2 2 3	Thm1
1 1 1 3 3 3 9 27	Thm1	1 1 2 2 2 4 9 16	Thm4	1 2 2 2 2 2 2 4	Thm1
1 1 1 3 3 3 18 18	Thm1	1 1 2 2 2 4 16 18	Thm1	1 2 2 2 2 2 2 6	Thm1
1 1 1 3 3 9 9 9	Thm1	1 1 2 2 2 4 16 20	Thm1	1 2 2 2 2 2 2 8	Thm1
1 1 1 3 3 9 9 12	Thm1	1 1 2 2 2 4 18 18	Thm1	1 2 2 2 2 2 2 9	Thm1
1 1 1 3 3 9 9 18	Thm1	1 1 2 2 2 5 5 9	Thm4	1 2 2 2 2 2 2 11	Thm9
1 1 1 3 3 9 9 21	Thm1	1 1 2 2 2 5 5 10	Thm4	1 2 2 2 2 2 3 8	Thm1
1 1 1 3 3 9 12 18	Thm1	1 1 2 2 2 5 9 10	Thm4	1 2 2 2 2 2 4 4	Thm1
1 1 1 3 6 9 9 9	Thm1	1 1 2 2 2 6 16 18	Thm1	1 2 2 2 2 2 4 5	Thm9
1 1 1 3 6 9 9 18	Thm1	1 1 2 2 2 8 8 8	Thm1	1 2 2 2 2 2 5 6	Thm9
1 1 1 4 4 4 4 4	Thm1	1 1 2 2 2 8 8 10	Thm1	1 2 2 2 2 2 5 8	Thm9
1 1 1 4 4 4 4 5	Thm1	1 1 2 2 2 8 8 16	Thm1	1 2 2 2 2 2 8 9	Thm4
1 1 2 2 2 2 2 2	Thm1	1 1 2 2 2 9 9 9	Thm1	1 2 2 2 2 2 8 10	Thm9
1 1 2 2 2 2 2 4	Thm1	1 1 2 2 2 9 9 10	Thm1	1 2 2 2 2 2 8 16	Thm4
1 1 2 2 2 2 2 5	Thm1	1 1 2 2 2 9 9 18	Thm1	1 2 2 2 2 2 9 9	Thm4
1 1 2 2 2 2 2 8	Thm1	1 1 2 2 2 9 9 19	Thm1	1 2 2 2 2 2 9 16	Thm4
1 1 2 2 2 2 2 9	Thm9	1 1 2 2 2 9 10 10	Thm4	1 2 2 2 2 2 9 18	Thm4
1 1 2 2 2 2 2 10	Thm1	1 1 2 2 2 9 10 18	Thm1	1 2 2 2 2 2 18 18	Thm1
1 1 2 2 2 2 2 18	Thm1	1 1 2 2 2 10 10 18	Thm1	1 2 2 2 2 3 3 4	Thm9
1 1 2 2 2 2 2 36	Thm1	1 1 2 2 2 10 10 20	Thm1	1 2 2 2 2 3 3 9	Thm9
1 1 2 2 2 2 3 4	Thm1	1 1 2 2 2 10 12 18	Thm1		

Table 12(cont): 8-Variable designs in order 48

8 – Variables	Ref	8 – Variables	Ref	8 – Variables	Ref
1 2 2 2 2 3 18 18	Thm1	2 2 2 2 2 2 4 6	Thm1	2 2 2 2 2 10 10 18	Thm1
1 2 2 2 2 4 4 4	Thm4	2 2 2 2 2 2 4 8	Thm9	2 2 2 2 3 3 16 18	Thm2
1 2 2 2 2 4 4 8	Thm4	2 2 2 2 2 2 4 16	Thm2	2 2 2 2 4 4 4 4	Thm4
1 2 2 2 2 4 8 8	Thm4	2 2 2 2 2 2 5 8	Thm9	2 2 2 2 4 4 4 8	Thm4
1 2 2 2 2 4 8 9	Thm4	2 2 2 2 2 2 8 9	Thm4	2 2 2 2 4 4 4 8	Thm4
1 2 2 2 2 4 9 16	Thm4	2 2 2 2 2 2 8 10	Thm9	2 2 2 2 4 4 8 8	Thm1
1 2 2 2 2 4 16 18	Thm1	2 2 2 2 2 2 9 9	Thm1	2 2 2 2 4 4 8 10	Thm1
1 2 2 2 2 4 16 19	Thm1	2 2 2 2 2 2 9 16	Thm4	2 2 2 2 4 4 8 16	Thm1
1 2 2 2 2 4 17 18	Thm1	2 2 2 2 2 2 9 18	Thm1	2 2 2 2 4 6 8 8	Thm1
1 2 2 2 2 5 5 9	Thm4	2 2 2 2 2 2 9 27	Thm1	2 2 2 2 4 8 8 8	Thm1
1 2 2 2 2 5 5 10	Thm4	2 2 2 2 2 2 16 18	Thm1	2 2 2 2 4 8 8 12	Thm1
1 2 2 2 2 5 9 10	Thm4	2 2 2 2 2 2 16 20	Thm1	2 2 2 2 4 8 8 18	Thm1
1 2 2 2 2 5 16 18	Thm1	2 2 2 2 2 2 18 18	Thm1	2 2 2 2 4 8 8 20	Thm1
1 2 2 2 2 8 8 8	Thm1	2 2 2 2 2 3 3 8	Thm1	2 2 2 2 4 8 10 18	Thm1
1 2 2 2 2 8 8 9	Thm1	2 2 2 2 2 4 4 4	Thm1	2 2 2 2 4 9 9 16	Thm1
1 2 2 2 2 9 9 9	Thm1	2 2 2 2 2 4 4 8	Thm1	2 2 2 2 4 9 9 18	Thm1
1 2 2 2 2 9 9 18	Thm1	2 2 2 2 2 4 4 10	Thm2	2 2 2 2 4 9 11 16	Thm1
1 2 2 2 2 9 10 10	Thm4	2 2 2 2 2 4 4 16	Thm1	2 2 2 2 5 5 10 18	Thm1
1 2 2 2 2 10 10 18	Thm1	2 2 2 2 2 4 8 8	Thm1	2 2 2 2 5 5 10 20	Thm1
1 2 2 2 2 10 10 19	Thm1	2 2 2 2 2 4 8 9	Thm4	2 2 2 2 5 5 12 18	Thm1
1 2 2 2 2 10 11 18	Thm1	2 2 2 2 2 4 8 12	Thm1	2 2 2 2 5 7 10 18	Thm1
1 2 2 2 3 3 3 8	Thm9	2 2 2 2 2 4 8 18	Thm1	2 2 2 2 6 6 10 18	Thm2
1 2 2 3 3 3 5 8	Thm9	2 2 2 2 2 4 8 26	Thm1	2 2 2 2 6 8 8 18	Thm1
1 2 2 2 3 4 16 18	Thm1	2 2 2 2 2 4 9 9	Thm1	2 2 2 2 6 9 9 16	Thm1
1 2 2 2 3 8 8 8	Thm1	2 2 2 2 2 4 9 16	Thm1	2 2 2 2 8 8 8 8	Thm1
1 2 2 2 3 10 10 18	Thm1	2 2 2 2 2 4 9 25	Thm1	2 2 2 2 9 9 10 10	Thm1
1 2 2 2 4 4 4 4	Thm4	2 2 2 2 2 4 16 18	Thm1	2 2 2 2 9 9 10 12	Thm1
1 2 2 2 4 4 4 8	Thm4	2 2 2 2 2 5 5 9	Thm4	2 2 2 2 9 10 10 11	Thm1
1 2 2 2 4 4 8 8	Thm4	2 2 2 2 2 5 5 10	Thm1	2 2 2 4 4 4 8 8	Thm1
1 2 2 2 4 8 8 8	Thm4	2 2 2 2 2 5 5 18	Thm1	2 2 2 4 4 8 8 8	Thm1
1 2 2 2 8 8 8 8	Thm1	2 2 2 2 2 5 5 28	Thm1	2 2 2 4 4 8 8 10	Thm1
1 2 2 2 8 8 8 9	Thm1	2 2 2 2 2 5 9 10	Thm4	2 2 2 4 4 8 8 18	Thm1
1 2 2 3 8 8 8 8	Thm1	2 2 2 2 2 5 10 18	Thm1	2 2 2 4 4 9 9 16	Thm1
1 2 3 3 3 9 9 9	Thm1	2 2 2 2 2 5 5 28	Thm1	2 2 2 4 5 5 10 18	Thm1
1 2 3 3 3 9 9 18	Thm1	2 2 2 2 2 5 9 10	Thm4	2 2 2 4 6 8 8 8	Thm1
1 3 3 3 3 3 3 3	Thm1	2 2 2 2 2 5 10 18	Thm1	2 2 2 4 9 9 10 10	Thm1
1 3 3 3 3 3 3 6	Thm1	2 2 2 2 2 5 10 23	Thm1	2 2 4 4 4 8 8 8	Thm1
1 4 4 4 4 4 4 4	Tab6	2 2 2 2 2 5 15 18	Thm1	3 3 3 3 3 3 3 3	Thm1
2 2 2 2 2 2 2 2	Thm1	2 2 2 2 2 8 8 8	Thm1	3 3 3 3 3 3 3 4	Thm1
2 2 2 2 2 2 2 4	Thm1	2 2 2 2 2 8 8 18	Thm1	3 3 3 3 3 3 3 6	Thm1
2 2 2 2 2 2 2 5	Thm1	2 2 2 2 2 8 8 22	Thm1	3 3 3 3 3 3 6 6	Thm4
2 2 2 2 2 2 2 8	Thm1	2 2 2 2 2 8 12 18	Thm1	3 3 3 3 3 6 6 6	Thm4
2 2 2 2 2 2 2 10	Thm9	2 2 2 2 2 9 9 10	Thm1	3 3 3 6 6 6 6 6	Thm4
2 2 2 2 2 2 2 16	Thm1	2 2 2 2 2 9 9 16	Thm1	3 3 6 6 6 6 6 6	Thm1
2 2 2 2 2 2 2 18	Thm1	2 2 2 2 2 9 9 18	Thm1	3 3 6 6 6 6 6 12	Thm1
2 2 2 2 2 2 2 34	Thm1	2 2 2 2 2 9 9 20	Thm1	3 6 6 6 6 6 6 6	Thm1
2 2 2 2 2 2 3 4	Thm1	2 2 2 2 2 9 10 10	Thm1	3 6 6 6 6 6 6 9	Thm1
2 2 2 2 2 2 3 8	Thm9	2 2 2 2 2 9 10 19	Thm1	5 5 5 5 5 5 5 5	Tab6
2 2 2 2 2 2 3 9	Thm9	2 2 2 2 2 9 11 18	Thm1	6 6 6 6 6 6 6 6	Thm1
2 2 2 2 2 2 4 4	Thm1	2 2 2 2 2 9 13 16	Thm1		

Table 12(cont): 8-Variable designs in order 48

7 – Variables	Ref	7 – Variables	Ref	7 – Variables	Ref
1 1 1 1 2 6 18	Thm9	1 1 2 3 3 3 3	16g	2 2 2 2 3 5 8	Thm9
1 1 1 1 4 6 18	Thm9	1 2 2 3 3 3 8	Thm9	2 2 2 3 3 3 8	Thm9
1 1 1 1 6 6 24	Thm9	1 2 2 2 3 6 8	Thm9	2 2 2 3 3 3 9	Thm9
1 1 1 2 4 10 10	Thm9	1 2 2 3 3 4 8	Thm9	2 2 2 3 3 4 8	Thm9
1 1 1 3 4 4 16	Thm9	1 2 2 3 3 5 8	Thm9	2 2 2 4 8 10 10	Thm9
1 1 1 4 4 6 6	Thm9	1 2 3 3 3 3 4	Thm9	2 2 3 3 3 3 8	Thm9
1 1 2 2 5 5 8	Thm9	2 2 2 2 2 6 8	Thm9		

Table 13: Some 7-Variable designs in order 48

6-Variables	Ref	6-Variables	Ref
1 1 1 2 4 10	XI	1 1 3 4 9 20	Thm9
1 1 1 3 4 9	XII	1 1 3 5 9 15	Thm9
1 1 1 3 4 18	Th2,XII	1 1 3 5 9 24	Thm9
1 1 1 4 4 20	Thm9	1 1 3 6 6 9	Thm9
1 1 1 4 5 20	Thm9	1 1 3 9 12 15	Thm9
1 1 1 4 6 6	XIII	1 1 4 4 6 9	Th2,XIV
1 1 1 4 6 9	Th2,XII	1 1 4 4 6 18	Th2,XIV
1 1 1 4 6 18	Th2,XII	1 1 4 4 9 9	Th2,III
1 1 1 4 8 20	Th2,XI	1 1 4 4 9 18	Th2,III
1 1 1 6 8 10	Th2,XII	1 1 4 6 8 9	Th2,XIV
1 1 1 6 8 20	Th2,XII	1 1 4 6 8 18	Th2,XIV
1 1 1 11 11 11	Th7,12jc	1 1 5 6 6 24	Thm9
1 1 2 2 8 20	Th2,XI	1 2 2 2 4 10	Th2,XI
1 1 2 2 8 25	Thm9	1 2 2 2 8 10	Th2,XI
1 1 2 2 9 18	Th2,III	1 2 2 2 8 18	Th4,IV
1 1 2 3 3 6	Th7,12fd	1 2 2 2 8 20	Th2,XI
1 1 2 3 4 9	Th2,XII	1 2 2 2 9 16	Th4,IV
1 1 2 3 4 9	XIV	1 2 2 2 9 18	Th2,III
1 1 2 3 4 18	Th2,XII	1 2 2 3 4 9	Th2,XII
1 1 2 3 8 18	Th2,XII	1 2 2 3 4 18	Th2,XII
1 1 2 4 4 10	Th2,XI	1 2 2 3 8 9	Th2,XII
1 1 2 4 4 20	Th2,XI	1 2 2 3 8 18	Th2,XII
1 1 2 4 6 9	Th2,XII	1 2 2 4 4 10	Th2,XI
1 1 2 4 6 18	Th2,XII	1 2 2 4 4 20	Th2,XI
1 1 2 4 8 10	Th2,XI	1 2 2 4 6 9	Th2,XII
1 1 2 4 8 20	Th2,XI	1 2 2 4 6 18	Th2,XIV
1 1 2 4 9 9	Th2,III	1 2 2 4 8 10	Th2,XI
1 1 2 4 9 18	Th2,III	1 2 2 4 8 18	Th4,IV
1 1 2 5 6 9	VI-4-7	1 2 2 4 8 20	Th2,XI
1 1 2 6 6 12	Th7,12fd	1 2 2 4 9 9	Th2,III
1 1 2 6 8 9	Th2,XIV	1 2 2 4 9 16	Th4,IV
1 1 2 6 8 10	Th2,XII	1 2 2 4 9 18	Th2,III
1 1 2 6 8 18	Th2,XIV	1 2 2 5 8 15	Thm9
1 1 2 6 8 20	Th2,XII	1 2 2 6 8 9	Th2,XIV
1 1 2 9 9 18	Th7,12hd	1 2 2 6 8 10	Th2,XIII
1 1 2 11 11 22	Th7,12jd	1 2 2 6 8 18	Th2,XIV
1 1 3 4 4 9	Th2,XIV	1 2 2 6 8 20	Th2,XIII
1 1 3 4 4 18	Th2,XIV	1 2 2 8 10 15	Thm9
1 1 3 4 8 9	Th2,XIV	1 2 2 8 12 15	Thm9
1 1 3 4 8 18	Th2,XIV	1 2 3 4 4 9	Th2,XIV

Table 14: Some 6-Variable designs in order 48

6-Variables	Ref	6-Variables	Ref
1 2 3 4 4 18	Th2,XIV	2 2 3 4 4 9	Th2,XIV
1 2 3 4 8 9	Th2,XIV	2 2 3 4 4 18	Th2,XIV
1 2 3 4 8 18	Th2,XIV	2 2 3 4 8 9	Th2,XIV
1 2 4 4 6 9	Th2,XIV	2 2 3 4 8 18	Th2,XIV
1 2 4 4 6 18	Th2,XIV	2 2 3 5 8 15	Thm9
1 2 4 4 8 9	Th4,IV	2 2 4 4 6 9	Th2,XIV
1 2 4 4 8 18	Th4,IV	2 2 4 4 6 18	Th2,XIV
1 2 4 4 9 9	Th2,III	2 2 4 4 8 9	Th4,IV
1 2 4 4 9 16	Th4,IV	2 2 4 4 9 9	Th2,III
1 2 4 4 9 18	Th2,III	2 2 4 4 9 16	Th4,IV
1 2 4 4 18 18	Th2,III	2 2 4 4 9 18	Th2,III
1 2 4 6 8 9	Th2,XIV	2 2 4 6 8 9	Th2,XIV
1 2 4 6 8 18	Th2,XIV	2 2 4 6 8 18	Th2,XIV
1 3 4 4 4 4	VII-3-4	2 2 4 5 5 10	Th7,12nd
1 3 3 5 9 12	Thm9	2 2 4 6 6 12	Th7,12od
1 4 4 4 8 9	Th4,IV	2 2 4 7 7 14	Th7,12pd
1 4 4 4 8 18	Th4,IV	2 2 5 8 12 15	Thm9
1 4 4 4 9 16	Th4,IV	2 3 5 5 6 9	Thm9
2 2 2 2 8 9	Th4,IV	2 3 5 5 9 12	Thm9
2 2 2 2 9 16	Th4,IV	2 4 4 4 8 9	Th4,IV
2 2 2 3 4 9	Th2,XIII	2 4 4 4 8 18	Th4,IV
2 2 2 3 4 18	Th2,XIII	2 4 4 4 9 16	Th4,IV
2 2 2 3 8 9	Th2,XIII	3 3 3 4 4 4	Th7,12uc
2 2 2 3 8 18	Th2,XIII	3 3 3 5 5 9	Thm9
2 2 2 4 6 9	Th2,XIII	3 3 3 7 7 7	Th7,12wc
2 2 2 4 6 18	Th2,XIII	3 3 3 8 8 8	Th7,12xc
2 2 2 4 8 9	Th4,IV	3 3 4 4 6 8	Th7,12ud
2 2 2 4 8 18	Th4,IV	3 3 6 7 7 14	Th7,12wd
2 2 2 4 9 9	Th2,III	3 3 6 8 8 16	Th7,12xd
2 2 2 4 9 16	Th4,IV	3 3 6 9 9 18	Th7,12yd
2 2 2 4 9 18	Th2,III	4 4 4 5 5 5	Th7,12Ac
2 2 2 5 5 5	Th7,12nc	4 4 4 6 6 6	Th7,12Bc
2 2 2 6 6 6	Th7,12oc	4 4 5 5 8 10	Th7,12Ad
2 2 2 6 8 9	Th2,XIV	4 4 6 6 8 12	Th7,12Bd
2 2 2 6 8 10	Th2,XIII	4 4 8 8 8 16	Th7,12Cd
2 2 2 6 8 18	Th2,XIV	5 5 5 6 6 6	Th7,12Ec
2 2 2 6 8 20	Th2,XIII	5 5 5 7 7 7	Th7,12Hc
2 2 2 7 7 7	Th7,12pc	5 5 6 6 10 12	Th7,12Ed
2 2 3 3 4 6	Th7,12 ℓ d	5 5 7 7 10 14	Th7,12Hd

Table 14 (cont): Some 6-Variable designs in order 48

5-Variables	Ref	5-Variables	Ref
1 3 3 3 3	Th7, 12cb	2 10 10 10 10	Th7, 12sb
1 5 5 5 5	Th7, 12eb	3 3 3 3 4	Th7, 12ua
1 6 6 6 6	Th7, 12fb	3 3 3 3 7	Th7, 12wa
1 9 9 9 9	Th7, 12hb	3 3 3 3 8	Th7, 12xa
1 10 10 10 10	Th7, 12ib	3 7 7 7 7	Th7, 12wb
1 11 11 11 11	Th7, 12jb	3 8 8 8 8	Th7, 12xb
2 2 2 2 7	Th7, 12pa	3 9 9 9 9	Th7, 12yb
2 3 3 3 3	Th7, 12 ℓ b	4 6 6 6 6	Th7, 12Bb
2 5 5 5 5	Th7, 12nb	5 5 5 5 6	Th7, 12Ec
2 6 6 6 6	Th7, 12ob	5 5 5 5 7	Th7, 12Ha
2 7 7 7 7	Th7, 12pb	5 6 6 6 6	Th7, 12Ed
2 9 9 9 9	Th7, 12rb	5 7 7 7 7	Th7, 12Hb

Table 15: 5-Variable designs in order 48 constructed using Construction 3

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